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Subleading jet functions in inclusive B decays

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Abstract: The contribution of subleading jet functions to inclusive decay distributions of B mesons are derived from a systematic two-step matching of QCD current correlators onto soft collinear and heavy quark effective theory. Focusing on the tree level matching of QCD onto soft collinear effective theory, the subleading jet functions are defined to all orders in $\alpha_s(\mu_i)$ (with $\mu_i^2 \sim m_b\Lambda_{\rm QCD}$) and are calculated explicitly at first order in $\alpha_s(\mu_i)$. We present explicit expressions for the decay rates of $\bar{B} \to X_u\, l\, \bar{\nu}$ and the $Q_{7\gamma} - Q_{7\gamma}$ contribution to $\bar{B} \to X_s\, \gamma$, where the subleading jet functions are multiplied by a tree level hard function and appear in a convolution with the leading order shape function. Together with the recent two loop calculation of the leading order hard function for $\bar{B} \to X_u\, l\, \bar{\nu}$, this paper will allow for a more precise description of inclusive B decays in the end point region.

Keywords: B-Physics, NLO Computations, Heavy Quark Physics, QCD

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Contents

1 Introduction

Charmless inclusive B decays play an important role in our understanding of the standard model and its possible extensions. The inclusive semileptonic B decay currently allows for the most accurate determination of $|V_{ub}|$, one of the fundamental parameters of the standard model, while the inclusive $\bar{B} \to X_s \gamma$ rate is used extensively in constraining models of new physics.

Since $\Lambda_{\rm QCD}$ is much smaller than the b-quark mass (m_b), one would expect that various physical observables for inclusive B decays can be expressed in terms of a local operator product expansion (OPE), where the various operators are suppressed by an increasing power of m_b. This is the case for the partial and total rate of $\bar{B} \to X_c\, l\, \bar{\nu}$, where schematically we have

$$d\Gamma \sim c_0\, O_0 + \sum_{i=2} \sum_j \frac{1}{m_b^i}\, c_i^j\, O_i{}^j. \tag{1.1}$$

The current state of the art is that c_0 is known at $\mathcal{O}(\alpha_s^2)$ [1, 2], while c_3^j [3] and c_4^j [4] are known at $\mathcal{O}(\alpha_s^0)$. For c_2^j, the coefficient of the "kinetic energy" operator c_2^1, is known at $\mathcal{O}(\alpha_s)$ [5], while c_2^2, the coefficient of the "chromomagnetic" operator, is known only at $\mathcal{O}(\alpha_s^0)$ [6, 7].

For the charmless inclusive B decays, $\bar{B} \to X_s\, \gamma$ and $\bar{B} \to X_u\, l\, \bar{\nu}$, the situation is more complicated. Experimental cuts force the hadronic jet X to have large energy $E_X \sim m_b$, but only moderate invariant mass $P_X^2 \sim m_b \Lambda_{\mathrm{QCD}}$. Consequently there are three energy scales in the problem: a hard scale ($\mu_h \sim m_b$), a hard-collinear scale ($\mu_i \sim \sqrt{m_b \Lambda_{\mathrm{QCD}}}$) and a soft scale ($\mu_s \sim \Lambda_{\mathrm{QCD}}$). For this kinematical region, often called the "end point region" or "the shape function region", the partial rates can be expressed in terms of a non-local OPE. Thus, for $\bar{B} \to X_u\, l\, \bar{\nu}$ and the $Q_{7\gamma} - Q_{7\gamma}$ contribution to $\bar{B} \to X_s\, \gamma$ we have an expansion which is schematically [8]:

$$\begin{aligned} d\Gamma_u &\sim H_u \cdot J \otimes S + \sum_{i=1} \sum_{j,k,l} \frac{1}{m_b^i}\, h_i^j \cdot j_i^k \otimes s_i^l \\ d\Gamma_s &\sim H_s \cdot J \otimes S + \sum_{i=1} \sum_{j,k,l} \frac{1}{m_b^i}\, h_i^j \cdot j_i^k \otimes s_i^l, \end{aligned} \tag{1.2}$$

where the hard functions (H, h_i) and the jet functions (J, j_i) are calculable in perturbation theory, while the shape functions (S, s_i) are non-local light-cone operators which are non perturbative objects.[1] Factorization theorems such as (1.2) are most conveniently proven using the Soft Collinear Effective Theory (SCET) [12–14].

The current state of the art is as follows. H_u was recently calculated at $\mathcal{O}(\alpha_s^2)$ [15–18] and H_s [19] is known[2] at $\mathcal{O}(\alpha_s)$. The leading order jet function J is known at $\mathcal{O}(\alpha_s^2)$ [25]. Of the $1/m_b$ corrections, only the terms of the form $h_1^0 \cdot j_1^0 \otimes s_1^1$ are explicitly known at $\mathcal{O}(\alpha_s^0)$ [26–28]. Therefore the factorization formula was proven only for the leading order term and one of the $1/m_b$ suppressed terms.

The knowledge of the leading order hard and jet functions at $\mathcal{O}(\alpha_s)$ and the $h_1^0 \cdot j_1^0 \otimes s_1^1$ terms at $\mathcal{O}(\alpha_s^0)$ (as well as known, but not properly factorized, α_s/m_b^i and $1/m_b^2$ corrections) was the basis of the precision determination of $|V_{ub}|$ in [29–31]. In order to improve the accuracy even further, one would like to know as much as possible about the properly factorized α_s^2, α_s/m_b, and the $1/m_b^2$ corrections, in decreasing order of importance. What would the calculation of these corrections entail?

In order to find the least important term, namely the $1/m_b^2$ corrections, the heavy-to-light SCET currents need to be matched at tree level to fourth order in the SCET

[1] For other contributions to $d\Gamma_s$, such as $Q_{7\gamma} - Q_{8g}$, one finds that more complicated factorization theorems hold [9–11]. We will not discuss these contributions in this paper.

[2] The $\mathcal{O}(\alpha_s^2)$ expression can be extracted from known results in the literature [20–23], as was done in [24] for a "normalized" H_s at the scale $\mu_h = m_b$.

expansion parameter $\sqrt{\Lambda_{\rm QCD}/m_b}$. The most important term, namely the α_s^2 corrections, should appear shortly now that the last ingredient, H_u, was calculated at $\mathcal{O}(\alpha_s^2)$ [15–18]. The second most important correction is of order α_s/m_b. From (1.2) one would naively expect to find three terms which scale like $1/m_b$ and need to be calculated to $\mathcal{O}(\alpha_s)$: $h^1 \cdot j^0 \otimes s^0, h^0 \cdot j^1 \otimes s^0$, and $h^0 \cdot j^0 \otimes s^1$ (the subscript 1 is implicit). Less formally, we expect to find subleading hard, jet and shape functions, respectively. Let us discuss each of these terms.

Since the hard functions are products of Wilson coefficients extracted in the matching of QCD onto SCET, they depend only on kinematical quantities which scale like m_b, i.e. m_b and $E_X + |\vec{P}_X|$. As such they always scale as $\mathcal{O}(1)$ in the $1/m_b$ expansion, so it is clear that terms of the form $h^1 \cdot j^0 \otimes s^0$ cannot appear at *any* order in α_s. In other words, subleading hard functions can appear only when they are multiplied by subleading jet or shape functions.

The "subleading shape functions" (SSF), i.e. terms of the form $h^0 \cdot j^0 \otimes s^1$ which arise already at $\mathcal{O}(\alpha_s^0)$, can be calculated, in principle, at $\mathcal{O}(\alpha_s)$, i.e. both h^0 and j^0 need to be calculated at $\mathcal{O}(\alpha_s)$. For the former a one loop matching of the SCET current to second order is needed, while the calculation of the latter was outlined (but not explicitly done!) in [26] and [28].

The focus of this paper is to prove that terms of the form $h^0 \cdot j^1 \otimes s^0$, i.e. "subleading jet functions" (SJF), indeed exist in the factorization formula for inclusive B decays. We will first prove the existence of such terms by showing that the partonic $\mathcal{O}(\alpha_s)$ terms in the hadronic tensor which are $1/m_b$ suppressed in the end point region arise from two momentum regions: soft and hard-collinear. We will then show how the soft region is accounted for by the parton level one loop diagrams of the *known* $\mathcal{O}(\alpha_s^0)$ $h^0 \cdot j^0 \otimes s^1$ term, and reproduce the hard-collinear region via time order products (TOPs) of subleading SCET currents. After establishing the need for the $h^0 \cdot j^1 \otimes s^0$ term, we will calculate the subleading jet functions via the usual two step matching. In the first step the QCD currents and Lagrangian are matched onto SCET at tree level and to second order in $\sqrt{\Lambda_{\rm QCD}/m_b}$. In the second step the SCET current correlator is matched onto Heavy Quark Effective Theory (HQET) [32] and the subleading jet functions are extracted. Since we are interested in α_s/m_b suppressed terms, and the subleading jet functions start at $\mathcal{O}(\alpha_s)$, it is sufficient to consider only the case of tree level matching of QCD onto SCET, for which we can use known results from the literature. For this case we will define the subleading jet functions to all orders in $\alpha_s(\mu_i)$, and calculate them explicitly at first order in $\alpha_s(\mu_i)$.

The subleading jet functions' contribution is, in some sense, the most important term at order α_s/m_b. When integrating over larger portions of phase space, the one loop subleading jet functions' contribution is no longer power suppressed. The other α_s/m_b term in the factorization formula, namely the $\mathcal{O}(\alpha_s)$ $h^0 \cdot j^0 \otimes s^1$ contribution, although formally α_s/m_b suppressed in the end point region, is expected to become even more power suppressed and thus is less important outside of the end point region. In other words, the terms which we will calculate are kinematically and not hadronically suppressed, and as such are important outside of the end point region. Incidentally, experiments are starting to probe the kinematic area outside of the end point region. Finally, since the subleading jet functions appear in convolution with the leading order shape function, their inclusion does not introduce new hadronic uncertainties.

Another motivation for studying subleading jet functions is that they arise also outside of the context of flavor physics. Since these functions encode the interaction of hard-collinear quarks and gluons, the same functions are expected to appear in the $x \to 1$ region of deep inelastic scattering [33–36].

The rest of the paper is organized as follows. After a short review of known results in section 2, we calculate in section 3 the partonic $\mathcal{O}(\alpha_s)$ terms in the hadronic tensor which are $1/m_b$ suppressed in the end point region. We then show that they arise from two momentum regions: soft and hard-collinear. In section 4 we explain how the soft region is accounted for by the parton level one loop diagrams of the *known* $h^0 \cdot j^0 \otimes s^1$ terms, and reproduce the hard-collinear region via time order products of subleading SCET currents. In the main section of the paper, section 5, we define the subleading jet functions, for the case of a tree level hard function, to all orders in $\alpha_s(\mu_i)$ and calculate their one loop expressions. After a short discussion of their renormalization, we present properly factorized expressions for the decay rates of $\bar{B} \to X_u \, l \, \bar{\nu}$ and the $Q_{7\gamma} - Q_{7\gamma}$ contribution to $\bar{B} \to X_s \, \gamma$. In section 6 we present our conclusions. The appendices contain proofs for some of the statements made in section 5. A reader who is mostly interested in the phenomenological results, can skip section 3 and 4 and proceed directly to section 5.

2 Review

In order to make the paper self-contained, we review in this section some known results about inclusive B decays, as well as some basic ingredients of SCET. For a more detailed account see [37].

Kinematical variables. The kinematics of inclusive B decays is such that in its rest frame, the B meson decays into a hadronic jet carrying momentum P_X and a non-hadronic jet (a lepton pair for $\bar{B} \to X_u \, l \, \bar{\nu}$ and a photon for $\bar{B} \to X_s \, \gamma$) carrying momentum q. Denoting by M_B the mass of the B meson and by v its four-velocity, we have therefore $M_B v = P_X + q$. Taking the four velocity of the B meson to be $v = (1,0,0,0)$ and $\vec{q}$ to point in the negative z direction, we define two light-like vectors $n^\mu = (1,0,0,1)$, $\bar{n}^\mu = (1,0,0,-1)$, such that $n + \bar{n} = 2v$, $n \cdot \bar{n} = 2$, and $n \cdot v = \bar{n} \cdot v = 1$. Any four vector a^μ can be decomposed as

$$a^\mu = \bar{n} \cdot a \frac{n^\mu}{2} + n \cdot a \frac{\bar{n}^\mu}{2} + a_\perp^\mu. \tag{2.1}$$

Notice that we have taken $v_\perp = q_\perp = 0$. Having fixed the two light-like vectors, rotational invariance implies that the transverse indices can only be contracted using

$$g_\perp^{\mu\nu} = g^{\mu\nu} - \frac{n^\mu \bar{n}^\nu + n^\nu \bar{n}^\mu}{2}, \qquad \epsilon_\perp^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} \bar{n}_\alpha n_\beta, \tag{2.2}$$

where $\epsilon_{0123} = 1$.

Conservation of 4-momentum implies that for the $\bar{B} \to X_s \, \gamma$ decay mode there is one independent kinematical variable, which we can take to be the photon energy E_γ or

$n \cdot P_X = M_B - 2E_\gamma$. The $\bar{B} \to X_u \, l \, \bar{\nu}$ decay mode has three independent variables which we can take to be [29, 38, 39]

$$P_+ = E_X - |\vec{P}_X| = n \cdot P_X, \quad P_- = E_X + |\vec{P}_X| = \bar{n} \cdot P_X, \quad P_l = M_B - 2E_l. \tag{2.3}$$

These are the "hadronic" variables. It is also useful to define a "partonic" set of variables. Let $\bar{\Lambda} = M_B - m_b$, where m_b is the b quark mass. Defining $p = m_b v - q = P_X - \bar{\Lambda} v$, we have $n \cdot p = n \cdot P - \bar{\Lambda}$, $\bar{n} \cdot p = \bar{n} \cdot P - \bar{\Lambda}$ as the corresponding partonic variables to P_+ and P_-. The hadronic tensor is naturally expressed in terms of $n \cdot p$ and $\bar{n} \cdot p$. Notice also that by construction $p_\perp = 0$, and as a result $p^2 = \bar{n} \cdot p \; n \cdot p$.

Hadronic tensor. Partial rates for the inclusive decays $\bar{B} \to X_u \, l \, \bar{\nu}$, and the $Q_{7\gamma} - Q_{7\gamma}$ contribution to $\bar{B} \to X_s \, \gamma$ can be calculated using the optical theorem. The central object to consider is the hadronic tensor, which is the discontinuity of a forward matrix element of a correlator of two currents:

$$W_{ij} = \frac{1}{\pi} \frac{1}{2M_B} \, \mathrm{Im} \left\langle \bar{B}(v) \left| i \int d^4x \, e^{iq\cdot x} \, T \left\{ J_i^\dagger(0) \, J_j(x) \right\} \right| \bar{B}(v) \right\rangle. \tag{2.4}$$

where, again, v is the velocity of the B-meson and q is the momentum of the lepton pair (photon) in the $\bar{B} \to X_u \, l \, \bar{\nu}$ ($\bar{B} \to X_s \, \gamma$) decay. The currents can generally be written as $J_i^\dagger = \bar{b} \, \Gamma_i \, q$ and $J_j = \bar{q} \, \Gamma_j \, b$. For semileptonic decays $\Gamma_i = \gamma^\mu (1 - \gamma_5)$ and $\Gamma_j = \gamma^\nu (1 - \gamma_5)$, and for the $Q_{7\gamma} - Q_{7\gamma}$ contribution to $\bar{B} \to X_s \, \gamma$, $\Gamma_i = \frac{1}{2}(1 - \gamma_5) \gamma_\mu^\perp \, \not{n}$, $\Gamma_j = \frac{1}{2} \not{n} \gamma_\perp^\mu (1 - \gamma_5)$. In order to somewhat simplify the traces in the expression for the hadronic tensor, we assume that Γ_i and Γ_j contain the same number of Dirac's gamma matrices, but otherwise we take $\Gamma_{i,j}$ to be arbitrary Dirac structures.

For radiative decays the hadronic tensor is given in term of one function $W \equiv W_{ij}$. The $Q_{7\gamma} - Q_{7\gamma}$ contribution to photon spectrum can then be written as[3]

$$\frac{d\Gamma}{dE_\gamma} = -\frac{G_F^2 \alpha}{4\pi^4} E_\gamma^3 |V_{tb} V_{ts}^*|^2 \, \overline{m}_b^2 \, |C_{7\gamma}^{\rm eff}|^2 \, W(P_+). \tag{2.5}$$

For semileptonic decays we can decompose the hadronic tensor in terms of five functions, $\tilde{W}_i(P_+, P_-)$,

$$\begin{aligned} W_{ij} = W^{\mu\nu} = & \, (n^\mu v^\nu + n^\nu v^\mu - g^{\mu\nu} - i\epsilon^{\mu\nu\alpha\beta} n_\alpha v_\beta) \tilde{W}_1 \\ & - g^{\mu\nu} \tilde{W}_2 + v^\mu v^\nu \tilde{W}_3 + (n^\mu v^\nu + n^\nu v^\mu) \tilde{W}_4 + n^\mu n^\nu \tilde{W}_5. \end{aligned} \tag{2.6}$$

The triple differential decay rate can be written in terms of $\tilde{W}_1, \ldots, \tilde{W}_5$ as [29]

$$\begin{aligned} \frac{d^3\Gamma}{dP_+ \, dP_- \, dP_l} = & \, \frac{G_F^2 |V_{ub}|^2}{16\pi^3} (M_B - P_+) \Big[(P_- - P_l)(M_B - P_- + P_l - P_+) \, \tilde{W}_1 \\ & + (M_B - P_-)(P_- - P_+) \frac{\tilde{W}_2}{2} + (P_- - P_l)(P_l - P_+) \left(\frac{y}{4} \tilde{W}_3 + \tilde{W}_4 + \frac{1}{y} \tilde{W}_5 \right) \Big], \end{aligned} \tag{2.7}$$

[3] See [29] for the exact definition of the various parameters in the this equation. Notice that W equals $-2U(\mu_h, \mu_i) \, \mathcal{F}_\gamma$ of [29].

where

$$y = \frac{P_- - P_+}{M_B - P_+}\,. \tag{2.8}$$

Known $1/m_b$ corrections. The hadronic tensor can be factorized as in equation (1.2). In this paper we will be interested in the terms which are suppressed by one power of m_b. There are currently two types of these terms which are known. The first type are "subleading shape functions" i.e. properly factorized terms of the form $h^0 \cdot j^0 \otimes s^1$, where h^0 and j^0 are explicitly known at $\mathcal{O}(\alpha_s^0)$. The second type are "kinematical power corrections", i.e. terms *calculated within the parton model* which are suppressed both by α_s and $1/m_b$. These terms will be properly factorized in this paper. We now briefly review these two types.

The contributions of the form $h^0 \cdot j^0 \otimes s^1$ to the hadronic tensor, i.e. the subleading shape functions, were calculated using SCET in [26–28] (for earlier partial calculations see [40–44]). Here we use the results of [27]. The above contribution to the hadronic tensor is:

$$\begin{aligned} W_{ij}^{\rm SSF} = & \int d\omega\, \delta(n\cdot p+\omega) \left[\frac{\omega\, S(\omega)+t(\omega)}{m_b}\, T_2 + \frac{s(\omega)}{m_b}\, T_1 + \frac{t(\omega)}{\bar n\cdot p}\, T_3 + \frac{u(\omega)}{\bar n\cdot p}\, T_1 - \frac{v(\omega)}{\bar n\cdot p}\, T_4 \right] \\ & -\pi\alpha_s \int d\omega\, \delta(n\cdot p+\omega) \left[\frac{f_u(\omega)}{\bar n\cdot p}\, T_1 + \frac{f_v(\omega)}{\bar n\cdot p}\, T_4 \right] \end{aligned} \tag{2.9}$$

where

$$\begin{aligned} T_1 &= \frac{1}{4}\,{\rm tr}\left[\Gamma_i\, \not{n}\, \Gamma_j\, \frac{1+\not{v}}{2}\right], & T_3 &= \frac{1}{4}\,{\rm tr}\left[\Gamma_i\, \gamma_\rho^\perp \gamma_5\, \Gamma_j\, \frac{1+\not{v}}{2}\, \gamma_\perp^\rho \gamma_5\right], \\ T_2 &= \frac{1}{8}\,{\rm tr}\left[\Gamma_i\, \not{n}\, \Gamma_j\, (\not{v}-\not{n})\right], & T_4 &= \frac{1}{4}\,{\rm tr}\left[\Gamma_i\, \not{n}\gamma_5\, \Gamma_j\, \frac{1+\not{v}}{2}\, (\not{v}-\not{n})\,\gamma_5\right]. \end{aligned} \tag{2.10}$$

The subleading shape functions are defined as:

$$\begin{aligned} \langle \bar h(0)\,[0,x_-]\,h(x_-)\rangle &= \int d\omega\, e^{-\frac{i}{2}\omega \bar n\cdot x}\, S(\omega)\,, \\ m_b\, \langle i \int d^4z\, T\{\bar h(0)\,[0,x_-]\,h(x_-)\mathcal{L}_h^{(2)}(z)\}\rangle &= \int d\omega\, e^{-\frac{i}{2}\omega \bar n\cdot x}\, s(\omega)\,, \\ \langle \bar h(0)\, \not{n}\,[0,x_-]\,(i\not{D}_\perp h)(x_-)\rangle &= \int d\omega\, e^{-\frac{i}{2}\omega \bar n\cdot x}\, t(\omega)\,, \\ -i \int_0^{\bar n\cdot x/2} dt\, \langle \bar h(0)\,[0,tn]\,(iD_\perp)^2(tn)\,[tn,x_-]\,h(x_-)\rangle &= \int d\omega\, e^{-\frac{i}{2}\omega \bar n\cdot x}\, u(\omega)\,, \\ -i \int_0^{\bar n\cdot x/2} dt\, \langle \bar h(0)\, \frac{\not{n}}{2}\,[0,tn]\,\sigma_{\mu\nu}^\perp\, gG_\perp^{\mu\nu}(tn)\,[tn,x_-]\,h(x_-)\rangle &= \int d\omega\, e^{-\frac{i}{2}\omega \bar n\cdot x}\, v(\omega)\,, \end{aligned} \tag{2.11}$$

and

$$2(-i)^2 \int\limits_0^{\bar{n}\cdot x/2} dt_1 \int\limits_{t_1}^{\bar{n}\cdot x/2} dt_2 \,\langle [\,(\bar{h}S)_0\, t_a]_k [t_a \left(S^\dagger h\right)_{x_-}]_l [\,(\bar{q}S)_{t_2 n}]_l \not{n} [\left(S^\dagger q\right)_{t_1 n}]_k\rangle$$
$$= \int d\omega\, e^{-\frac{i}{2}\omega \bar{n}\cdot x}\, f_u(\omega)\,,$$
$$2(-i)^2 \int\limits_0^{\bar{n}\cdot x/2} dt_1 \int\limits_{t_1}^{\bar{n}\cdot x/2} dt_2 \,\langle [\,(\bar{h}S)_0\, t_a]_k \not{n}\gamma_5 [t_a \left(S^\dagger h\right)_{x_-}]_l [\,(\bar{q}S)_{t_2 n}]_l \not{n}\gamma_5 [\left(S^\dagger q\right)_{t_1 n}]_k\rangle$$
$$= \int d\omega\, e^{-\frac{i}{2}\omega \bar{n}\cdot x}\, f_v(\omega)\,, \tag{2.12}$$

where k, l are color indices, S in equation (2.12) is a soft Wilson line defined in [27] (not to be confused with the leading order shape function $S(\omega)$!), $[x, y] \equiv S(x)S^\dagger(y)$, $\mathcal{L}_h^{(2)}$ is the next-to-leading term in the expansion of the HQET Lagrangian, and

$$\langle \bar{h} \dots h\rangle \equiv \frac{\langle \bar{B}(v)|\, \bar{h} \dots h\, |\bar{B}(v)\rangle}{2M_B}.$$

The second type of terms, namely the kinematical power corrections, can be found in [29], where they were called $\mathcal{F}^{\rm kin}$. In that paper the corrections were convoluted with the "tree level shape function", in absence of proper factorization. The relevant expressions are, for the $Q_{7\gamma} - Q_{7\gamma}$ contribution to $\bar{B} \to X_s\, \gamma$,

$$W = -\frac{2}{M_B - P_+} \frac{C_F\, \alpha_s(\bar{\mu})}{4\pi} \int_0^{P_+} d\hat{\omega}\, \hat{S}(\hat{\omega}, \mu_i)\, (-15 - 16\, \ln x)\,, \tag{2.13}$$

and for $\bar{B} \to X_u\, l\, \bar{\nu}$,

$$\begin{aligned}
\tilde{W}_1^{\rm kin(1)} &= \frac{1}{M_B - P_+} \frac{C_F \alpha_s(\bar{\mu})}{4\pi} \\
&\quad \int_0^{P_+} d\hat{\omega}\, \hat{S}(\hat{\omega}, \mu_i) \left[6 - \frac{5}{y} + \left(\frac{12}{y} - 4\right) \ln \frac{y}{x}\right], \\
\tilde{W}_2^{\rm kin(1)} &= \frac{1}{M_B - P_+} \frac{C_F \alpha_s(\bar{\mu})}{4\pi} \int_0^{P_+} d\hat{\omega}\, \hat{S}(\hat{\omega}, \mu_i) \left[\frac{2}{y}\right], \\
\left(\frac{y}{4}\tilde{W}_3 + \tilde{W}_4 + \frac{1}{y}\tilde{W}_5\right)^{\rm kin(1)} &= \frac{1}{M_B - P_+} \frac{C_F \alpha_s(\bar{\mu})}{4\pi} \int_0^{P_+} d\hat{\omega}\, \hat{S}(\hat{\omega}, \mu_i) \left[4 - \frac{22}{y} + \frac{8}{y} \ln \frac{y}{x}\right].
\end{aligned} \tag{2.14}$$

where

$$x = \frac{P_+ - \hat{\omega}}{M_B - P_+}.$$

The "hatted" function $\hat{S}(\hat{\omega}, \mu_i)$ is related to $S(\omega)$, defined in equation (2.11), by a change of variables and a $1/m_b$ suppressed term. The exact relation can be found in [29].

JHEP06(2009)083

Since the expansion in [29] was organized in inverse powers of $M_B - P_+$ instead of m_b, for future reference we will need also the leading order term for $\tilde{W}_1$

$$\tilde{W}_1^{(0)} = H_1(y, \mu_h) \int_0^{P_+} d\hat{\omega}\, y m_b\, J(y m_b(P_+ - \hat{\omega}), \mu_i)\, \hat{S}(\hat{\omega}, \mu_i)\,, \tag{2.15}$$

where

$$\begin{aligned} H_{u1}(y, \mu_h) \;=\; 1 + \frac{C_F \alpha_s(\mu_h)}{4\pi} \bigg[& -4\ln^2 \frac{y m_b}{\mu_h} + 10 \ln \frac{y m_b}{\mu_h} - 4 \ln y \\ & -\frac{2 \ln y}{1-y} - 4 L_2(1-y) - \frac{\pi^2}{6} - 12 \bigg], \end{aligned} \tag{2.16}$$

and

$$J(p^2, \mu) = \delta(p^2) \left[1 + \frac{C_F \alpha_s(\mu)}{4\pi} (7 - \pi^2) \right] + \frac{C_F \alpha_s(\mu)}{4\pi} \left[\frac{1}{p^2} \left(4 \ln \frac{p^2}{\mu^2} - 3 \right) \right]_*^{[\mu^2]}. \tag{2.17}$$

Some ingredients of SCET. SCET is the appropriate effective field theory to discuss inclusive B decays in the end point region. The SCET expansion parameter is $\sqrt{\lambda} \equiv \sqrt{\Lambda_{\rm QCD}/m_b}$. For the following we will need the expansion of the SCET heavy-to-light currents to second order, as well as the leading order hard-collinear Lagrangian.[4] These are most conveniently listed in [27].

First, the hard-collinear Lagrangian is

$$\begin{aligned} \mathcal{L}_\xi^{(0)} &= \bar{\xi}^{(0)} \frac{\not{\bar{n}}}{2} \left(i n \cdot D_{hc}^{(0)} + i \not{D}_{\perp hc}^{(0)} \frac{1}{i \bar{n} \cdot D_{hc}^{(0)}} i \not{D}_{\perp hc}^{(0)} \right) \xi^{(0)} \\ &= \bar{\xi}^{(0)} \frac{\not{\bar{n}}}{2} \left(i n \cdot D_{hc}^{(0)} + i \not{D}_{\perp hc}^{(0)}\, W \frac{1}{i \bar{n} \cdot \partial} W^\dagger\, i \not{D}_{\perp hc}^{(0)} \right) \xi^{(0)}\,, \end{aligned} \tag{2.18}$$

where the Lagrangian is written in terms of "sterile" fields, i.e. after the soft degrees of freedom were decoupled via a field redefinition. In the last equation $i D_{hc}^{(0)\mu} = i\partial^\mu + g A_{hc}^{(0)\mu}$ and

$$W = P \exp \left(ig \int_{-\infty}^{0} dt\, \bar{n} \cdot A_{hc}^{(0)}(x + t\bar{n}) \right). \tag{2.19}$$

Next we need the expressions for the currents. We present them in terms of the "calligraphic fields" [45]

$$\mathcal{X} = W^\dagger \xi^{(0)}\,, \qquad \mathcal{A}_{hc}^\mu = W^\dagger (i D_{hc}^{(0)\mu} W). \tag{2.20}$$

Generally speaking, in the SCET expansion of the currents (and the Lagrangian), the power suppression arises from two separate sources, not mutually exclusive: presence of power suppressed components of the hard-collinear gluon field or the hard-collinear

[4] The power corrections to the hard-collinear Lagrangian always involve extra soft particles (apart from the heavy quark) and as such do not contribute to the subleading jet functions.

covariant derivative, and presence of soft fields or their covariant derivatives. For the purpose of this paper we need only currents of the first type, which are

$$
\begin{aligned}
J^{(0)} &= \bar{\mathfrak{X}}\,\Gamma\left(S^\dagger h\right)_{x_-}\,, \\
J^{(1)} &= -\,\bar{\mathfrak{X}}\,\frac{\bar{\not{n}}}{2}\,\not{\mathcal{A}}_{\perp hc}\,\frac{1}{i\bar{n}\cdot\overleftarrow{\partial}}\,\Gamma\left(S^\dagger h\right)_{x_-} - \bar{\mathfrak{X}}\,\Gamma\,\frac{\not{n}}{2m_b}\,\not{\mathcal{A}}_{\perp hc}\left(S^\dagger h\right)_{x_-}\,, \\
J^{(2)} &= -\,\bar{\mathfrak{X}}\,\Gamma\frac{\not{n}}{2m_b}\,n\cdot\mathcal{A}_{hc}\left(S^\dagger h\right)_{x_-} - \bar{\mathfrak{X}}\,\Gamma\,\frac{1}{i\bar{n}\cdot\partial}\,n\cdot\mathcal{A}_{hc}\left(S^\dagger h\right)_{x_-} \\
&\quad -\,\bar{\mathfrak{X}}\,\Gamma\,\frac{1}{i\bar{n}\cdot\partial}\,\frac{(i\not{\mathcal{D}}_{\perp hc}\not{\mathcal{A}}_{\perp hc})}{m_b}\left(S^\dagger h\right)_{x_-} + \bar{\mathfrak{X}}\,\frac{i\overleftarrow{\not{\mathcal{D}}}_{\perp hc}}{m_b}\,\frac{1}{i\bar{n}\cdot\overleftarrow{\partial}}\,\frac{\bar{\not{n}}}{2}\,\Gamma\,\frac{\not{n}}{2}\,\not{\mathcal{A}}_{\perp hc}\left(S^\dagger h\right)_{x_-}
\end{aligned}
\tag{2.21}
$$

where we have suppressed the overall $e^{-im_b v\cdot x}$ factor in each term. For completeness we list also the currents of the second type, which we will not use

$$
\begin{aligned}
J^{(1)}_{\text{not used}} &= \bar{\mathfrak{X}}\,\Gamma\,x_\perp^\mu\left(S^\dagger D_\mu h\right)_{x_-} + \bar{\mathfrak{X}}\,\frac{\bar{\not{n}}}{2}\,i\overleftarrow{\not{\partial}}_\perp\,\frac{1}{i\bar{n}\cdot\overleftarrow{\partial}}\,\Gamma\left(S^\dagger h\right)_{x_-}\,, \\
J^{(2)}_{\text{not used}} &= \bar{\mathfrak{X}}\,\Gamma\left[\frac{n\cdot x}{2}\left(S^\dagger\bar{n}\cdot Dh\right)_{x_-} + \frac{x_\perp^\mu x_\perp^\nu}{2}\left(S^\dagger D_\mu D_\nu h\right)_{x_-} + \left(S^\dagger\,\frac{i\not{D}}{2m_b}\,h\right)_{x_-}\right] \\
&\quad +\bar{\mathfrak{X}}\,\frac{\bar{\not{n}}}{2}\,i\overleftarrow{\not{\partial}}_\perp\,\frac{1}{i\bar{n}\cdot\overleftarrow{\partial}}\,\Gamma\,x_\perp^\mu\left(S^\dagger D_\mu h\right)_{x_-} \\
&\quad -\bar{\mathfrak{X}}\left(\frac{\bar{\not{n}}}{2}\,\not{\mathcal{A}}_{\perp hc}\,\frac{1}{i\bar{n}\cdot\overleftarrow{\partial}}\,\Gamma + \Gamma\,\frac{\not{n}}{2m_b}\,\not{\mathcal{A}}_{\perp hc}\right)x_\perp^\mu\left(S^\dagger D_\mu h\right)_{x_-}\,.
\end{aligned}
\tag{2.22}
$$

The second term in $J^{(1)}_{\text{not used}}$ does not contain any soft fields apart from the heavy quark. Still, we can ignore its contribution in this paper, since we can always set $p_\perp$, where p is the total hard-collinear momentum, to zero.

It should be noted that these currents were matched at zeroth order in $\alpha_s(\mu_h)$ and as a result the Wilson coefficients are always equal to 1. When matching beyond zeroth order in $\alpha_s(\mu_h)$, one would expect more complicated currents with multiple non-localities, see for example the one loop matching onto the first order SCET current in [46]. The resulting contributions to the hadronic tensor would be suppressed by $\alpha_s(\mu_h)\times\alpha_s(\mu_i)\times 1/m_b$ and as such are much smaller than the contributions considered in this paper.

3 Analysis by regions

In this section we will calculate, within the parton model, the one loop corrections to the hadronic tensor, defined in equation (2.4), that scale as $\mathcal{O}(\lambda^0)$ in the shape-function region, i.e. terms which are suppressed by $1/m_b$ compared to the leading order terms, which scale as $\mathcal{O}(\lambda^{-1})$. We perform the calculation for a general Dirac structure in the hadronic tensor. In order to simplify the calculation we neglect the residual momentum of the b quark, i.e. we work with on-shell b quarks. As a result, all the terms are constants or logarithms of the form $\ln{(n\cdot p)}/(\bar{n}\cdot p)\equiv\ln r$, where p is the partonic momentum of the jet, defined in section 2, which satisfies $p_\perp = 0$ and $p^2 = \bar{n}\cdot p\; n\cdot p$.

JHEP06(2009)083

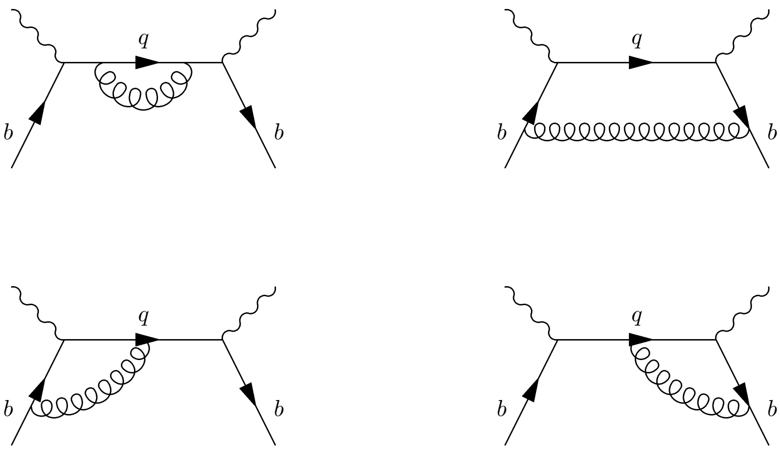


Figure 1. One loop diagrams contributing to the hadronic tensor, top left: "Self energy" diagram, top right: "Box" diagram, bottom line: two "Vertex" diagrams. The letter next to each solid line denotes the flavor of the quark.

We perform the calculation both in "full QCD" and by using the method or regions [47, 48]. We find that only two kinematical regions are needed: a hard-collinear region, where the loop momentum scales as $(1, \lambda, \lambda^{1/2})$, and a soft region, where the loop momentum scales as $(\lambda, \lambda, \lambda)$. There is no contribution from a hard region, were the loop momentum scales as $(1, 1, 1)$, which is in accordance with the lack of terms of the form $h^1 \cdot j^0 \otimes s^0$ in the factorization formula. As explained in the introduction this is a general result which holds also beyond one loop order. Typically, we find that terms of the form $\ln r$ can be decomposed as:

$$\ln r \equiv \ln \frac{n \cdot p}{\bar{n} \cdot p} = \underbrace{-\frac{1}{\epsilon} + \ln \frac{\mu^2}{p^2}}_{\text{hard}-\text{collinear}} + \underbrace{\frac{1}{\epsilon} + \ln \frac{(n \cdot p)^2}{\mu^2}}_{\text{soft}} . \tag{3.1}$$

At one loop there are several diagrams that contribute to the hadronic tensor. These diagrams are shown in figure 1. We present the results for each diagram separately, namely, the "Self energy" diagram (top left), the "Box" diagram (top right), and the "Vertex" contribution which is the *sum* of the diagrams on the bottom line of figure 1. We use Feynman gauge throughout this paper.

3.1 Full QCD

Calculating the diagrams we find the following results:

- Self energy:

$$W_{ij} = \frac{C_F \alpha_s}{4\pi} \, \theta(p^2) \, \mathrm{tr} \left[\Gamma_i \, \bar{n}\!\!\!/ \, \Gamma_j \, \frac{1 + v\!\!\!/}{2} \right] \frac{1}{\bar{n} \cdot p} \, \frac{1}{4}, \tag{3.2}$$

- Box:

$$W_{ij} = \frac{C_F\alpha_s}{4\pi}\,\theta(p^2)\left\{\mathrm{tr}\left[\Gamma_i\,\not{n}\,\Gamma_j\,\frac{1+\not{v}}{2}\right]\,\frac{1}{\bar{n}\cdot p}\,(-1-\ln r)+\mathrm{tr}\left[\Gamma_i\,\not{n}\,\Gamma_j\,\not{\bar{n}}\right]\,\frac{1}{m_b}\,\frac{1}{4}\right.$$
$$\left.+\mathrm{tr}\left[\Gamma_i\,\not{n}\,\Gamma_j\,\not{n}\right]\,\frac{1}{m_b}\,\frac{1}{4}\,(-2-\ln r)+\mathrm{tr}\left[\Gamma_i\,\not{n}\,\Gamma_j\,\not{n}\right]\,\frac{\bar{n}\cdot p}{m_b^2}\,\frac{1}{16}\right\}, \tag{3.3}$$

- Vertex:

$$W_{ij} = \frac{C_F\alpha_s}{4\pi}\,\theta(p^2)\left\{\mathrm{tr}\left[\Gamma_i\,\not{n}\,\Gamma_j\,\not{\bar{n}}\right]\,\frac{1}{m_b}\,\frac{1}{4}\left(\frac{3}{2}+\ln r\right)\right.$$
$$-\mathrm{tr}\left[\Gamma_i\,\not{n}\,\Gamma_j\,\frac{1+\not{v}}{2}\right]\,\frac{1}{\bar{n}\cdot p}\,(1+2\ln r)$$
$$-\left(\mathrm{tr}\left[\Gamma_i\,\not{n}\,\gamma^\beta\,\gamma_\perp^\alpha\,\Gamma_j\,\gamma_\alpha^\perp\,\gamma_\beta\,\frac{1+\not{v}}{2}\right]+\mathrm{tr}\left[\Gamma_i\,\gamma_\perp^\alpha\,\gamma^\beta\,\not{n}\,\Gamma_j\,\frac{1+\not{v}}{2}\,\gamma_\beta\,\gamma_\alpha^\perp\right]\right)\,\frac{1}{m_b}\,\frac{1}{16}$$
$$\left.-\mathrm{tr}\left[\Gamma_i\,\gamma_\perp^\alpha\,\Gamma_j\,\gamma_\alpha^\perp\right]\,\frac{1}{m_b}\,\frac{1}{8}\right\}. \tag{3.4}$$

In order to check these results, we can compare them to the expansion of the one loop expressions of the hadronic tensor for $\bar{B}\to X_u\,l\,\bar{\nu}$, and the $Q_{7\gamma}-Q_{7\gamma}$ contribution to $\bar{B}\to X_s\,\gamma$. This is most easily done by using equations (2.13)–(2.17) in section 2, which are taken from [29]. In that paper the correction were convoluted with the "tree level shape function". We can undo this convolution by the replacements:

$$M_B - P_+ \to m_b,\qquad y\to\frac{\bar{n}\cdot p}{m_b},\qquad \frac{x}{y}\to\frac{n\cdot p}{\bar{n}\cdot p},\qquad \int_0^{P_+} d\hat{\omega}\,\hat{S}(\hat{\omega},\mu_i)\to 1. \tag{3.5}$$

For $\bar{B}\to X_u\,l\,\bar{\nu}$ we also need to expand $\tilde{W}_1^{(0)}$ in powers of $n\cdot p/\bar{n}\cdot p$. In total we find for $\bar{B}\to X_s\,\gamma$:

$$W = -\frac{C_F\alpha_s}{4\pi}\frac{2}{m_b}\,(-15-16\,\ln r)\,, \tag{3.6}$$

and for $\bar{B}\to X_u\,l\,\bar{\nu}$:

$$\tilde{W}_1 = \frac{C_F\alpha_s}{4\pi}\left(\frac{10}{m_b}-\frac{9}{\bar{n}\cdot p}-\frac{12}{\bar{n}\cdot p}\ln r+\frac{4}{m_b}\ln r\right)$$
$$\tilde{W}_2 = \frac{C_F\alpha_s}{4\pi}\frac{2}{\bar{n}\cdot p}$$
$$\frac{y}{4}\tilde{W}_3+\tilde{W}_4+\frac{1}{y}\tilde{W}_5 = \frac{C_F\alpha_s}{4\pi}\left(\frac{4}{m_b}-\frac{22}{\bar{n}\cdot p}-\frac{8}{\bar{n}\cdot p}\ln r\right). \tag{3.7}$$

Summing over (3.2), (3.3), and (3.4), and calculating the traces for each decay mode using the expressions after equation (2.4), we find complete agreement with (3.6) and (3.7).

We are now ready to repeat this calculation using the method of regions. We perform the calculation in $d=4-2\epsilon$ dimensions and use dimensional regularization to regularize both the UV and IR divergences. We also implicitly take $\mu\to\mu e^{\gamma_E/2}(4\pi)^{-1/2}$.

3.2 Hard-collinear region

For the hard-collinear region we find the following results.

- Self energy:

$$W_{ij} = \theta(p^2)\frac{C_F\alpha_s}{4\pi}\,\mathrm{tr}\left[\Gamma_i\,\not{\bar{n}}\,\Gamma_j\,\frac{1+\not{v}}{2}\right]\,\frac{1}{\bar{n}\cdot p}\,\frac{1}{4}, \tag{3.8}$$

- Box:

$$\begin{aligned} W_{ij} &= \theta(p^2)\frac{C_F\alpha_s}{4\pi}\Bigg\{\mathrm{tr}\left[\Gamma_i\,\not{n}\,\Gamma_j\,\frac{1+\not{v}}{2}\right]\,\frac{1}{\bar{n}\cdot p}\,\left(-\frac{1}{\epsilon}+2-\ln\frac{\mu^2}{p^2}\right) \\ &\quad +\mathrm{tr}\left[\Gamma_i\,\not{n}\,\Gamma_j\,\not{n}\right]\,\frac{1}{m_b}\,\frac{1}{4}\,\left(-\frac{1}{\epsilon}-1-\ln\frac{\mu^2}{p^2}\right)+\mathrm{tr}\left[\Gamma_i\,\not{n}\,\Gamma_j\,\not{n}\right]\,\frac{\bar{n}\cdot p}{m_b^2}\,\frac{1}{16}\Bigg\}, \end{aligned} \tag{3.9}$$

- Vertex:

$$\begin{aligned} W_{ij} &= \theta(p^2)\frac{C_F\alpha_s}{4\pi}\Bigg\{\mathrm{tr}\left[\Gamma_i\,\not{n}\,\Gamma_j\,\frac{1+\not{v}}{2}\right]\,\frac{2}{\bar{n}\cdot p}\,\left(-\frac{1}{\epsilon}+\frac{1}{2}-\ln\frac{\mu^2}{p^2}\right)+ \\ &\quad +\mathrm{tr}\left[\Gamma_i\,\not{n}\,\Gamma_j\,\not{\bar{n}}\right]\,\frac{1}{m_b}\,\frac{1}{4}\left(\frac{1}{\epsilon}+\frac{3}{2}+\ln\frac{\mu^2}{p^2}\right)-\mathrm{tr}\left[\Gamma_i\,\gamma_\perp^\alpha\,\Gamma_j\,\gamma_\alpha^\perp\right]\frac{1}{m_b}\,\frac{1}{8} \\ &\quad -\left(\mathrm{tr}\left[\Gamma_i\,\not{n}\,\gamma^\beta\,\gamma_\perp^\alpha\,\Gamma_j\,\gamma_\alpha^\perp\,\gamma_\beta\,\frac{1+\not{v}}{2}\right]+\mathrm{tr}\left[\Gamma_i\,\gamma_\perp^\alpha\,\gamma^\beta\,\not{n}\,\Gamma_j\,\frac{1+\not{v}}{2}\,\gamma_\beta\,\gamma_\alpha^\perp\right]\right)\,\frac{1}{m_b}\,\frac{1}{16}\Bigg\}. \end{aligned} \tag{3.10}$$

3.3 Soft region

For the soft region we find that the "self energy" diagram does not contribute. For the other diagrams we have:

- Box

$$\begin{aligned} W_{ij} &= \theta(n\cdot p)\frac{C_F\alpha_s}{4\pi}\Bigg\{\mathrm{tr}\left[\Gamma_i\,\not{n}\,\Gamma_j\,\frac{1+\not{v}}{2}\right]\,\frac{1}{\bar{n}\cdot p}\,\left(\frac{1}{\epsilon}-3-\ln\frac{(n\cdot p)^2}{\mu^2}\right) \\ &\quad +\mathrm{tr}\left[\Gamma_i\,\not{n}\,\Gamma_j\,\not{\bar{n}}\right]\,\frac{1}{m_b}\,\frac{1}{4}+\mathrm{tr}\left[\Gamma_i\,\not{n}\,\Gamma_j\,\not{n}\right]\frac{1}{m_b}\,\frac{1}{4}\left(\frac{1}{\epsilon}-1-\ln\frac{(n\cdot p)^2}{\mu^2}\right)\Bigg\}, \end{aligned} \tag{3.11}$$

- Vertex

$$\begin{aligned} W_{ij} &= \theta(n\cdot p)\frac{C_F\alpha_s}{4\pi}\Bigg\{\mathrm{tr}\left[\Gamma_i\,\not{n}\,\Gamma_j\,\frac{1+\not{v}}{2}\right]\,\frac{1}{\bar{n}\cdot p}\,2\left(\frac{1}{\epsilon}-1-\ln\frac{(n\cdot p)^2}{\mu^2}\right)+ \\ &\quad +\mathrm{tr}\left[\Gamma_i\,\not{n}\,\Gamma_j\,\not{\bar{n}}\right]\,\frac{1}{m_b}\,\frac{1}{4}\left(-\frac{1}{\epsilon}+\ln\frac{(n\cdot p)^2}{\mu^2}\right)\Bigg\} \end{aligned} \tag{3.12}$$

Adding up the two types of regions we find that, as expected, the sum of the hard-collinear and soft regions reproduce the full QCD result. Notice also that the structure of the soft region is simpler than that of the hard-collinear region. In the next section we will see that the reason is that the soft region is accounted for by the parton level one loop expressions for only two subleading shape function, while for the hard-collinear region we need several subleading jet functions.

4 Effective field theory calculation

We have seen in the previous section that the one loop corrections to the hadronic tensor which scale as $\mathcal{O}(\lambda^0)$ in the end point region, arise from two kinematical regions: a hard-collinear region and a soft region. In this section we will see how the soft region is accounted for by the one loop corrections to the subleading shape functions' contribution calculated within the parton model. The hard-collinear region is accounted for by the contribution of the time ordered product of power suppressed SCET currents. This calculation is the basis for the subleading jet function calculation which we perform in the next section.

4.1 Soft contribution

The contribution of the soft region can be fully accounted for by calculating, within the parton model, the one loop corrections to the subleading shape functions. In particular there is no need to introduce new subleading shape functions. From equation (2.9) we see that the hadronic tensor depends on several subleading shape functions. Naively, one would assume that we need to calculate the one loop corrections for $\omega S(\omega), s(\omega), t(\omega), u(\omega), v(\omega)$, as well as the four-quark shape functions $f_u(\omega), f_v(\omega)$. In practice, only the contributions of $\omega S(\omega)$ and $u(\omega)$ are needed for the following reasons.

- We have chosen the coordinate system such that $v_\perp = 0$. As a result the matrix elements of the operator corresponding to $t(\omega)$ and $v(\omega)$, vanish at one loop, since they only contain gluons which have perpendicular polarization.

- The matrix elements of the operators corresponding to $f_u(\omega)$ and $f_v(\omega)$ vanish since they involve scaleless integrals over the $\bar{n}$ component of the light quark momentum.

- Setting the residual momentum of the heavy quarks to zero, we find that the matrix element of the operator corresponding to $s(\omega)$ vanishes.

As a result we need the "zero external gluon" matrix elements of the operators corresponding to $\omega S(\omega)$ and $u(\omega)$. The first can be extracted from [38]. After setting the residual momentum k to zero, we find

$$\omega\, S_{\rm bare}^{\rm parton} = \theta(-\omega)\, \frac{C_F \alpha_s}{\pi} \left(-\frac{1}{\epsilon} + \ln\frac{\omega^2}{\mu^2} + 1\right). \tag{4.1}$$

For $u(\omega)$, we need to calculate the one loop amplitude which is the sum of the diagrams shown in figure 2. The relevant Feynman rules needed for this calculation involve zero and

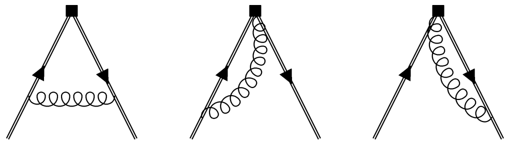
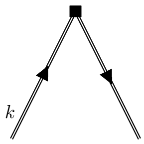
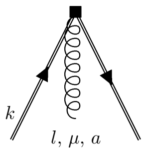


Figure 2. One loop diagrams contributing to the parton level expression of $u(\omega)$.

one external gluon. We have



$$-k_\perp^2\,\delta(\omega - n\cdot k)$$



$$\frac{t^a\,g\,n^\mu}{n\cdot l}\Big[(k-l)_\perp^2\,\delta'(\omega - n\cdot k + n\cdot l) - k_\perp^2\,\delta'(\omega - n\cdot k)\Big]$$

$$+\,\frac{t^a\,g\,n^\mu}{(n\cdot l)^2}(2k_\perp\cdot l_\perp - l_\perp^2)\Big[\delta(\omega - n\cdot k + n\cdot l) - \delta(\omega - n\cdot k)\Big]. \quad (4.2)$$

The notation is such that k is the incoming heavy quark residual momentum and l, μ, a are the outgoing gluon's momentum, polarization, and color index, respectively. For the one gluon Feynman rule we have omitted terms in which the gluon has a perpendicular polarization, since such terms do not contribute to the one loop amplitude for $v_\perp = 0$.

These Feynman rules were first calculated by Trott and Williamson in [49], where the corresponding operator is called $Q_3(\omega,\Gamma)$. Keeping only the terms which are proportional to n^μ in the one external gluon Feynman rule of [49], our result agrees with theirs accounting for an overall factor of $2/m_b$ and a different sign of the heavy quark residual momentum, both arising from the slightly different definition of $u(\omega)$. Calculating the one loop amplitude, which was not calculated in [49], and setting the residual momentum k to zero, we find

$$u_{\rm bare}^{\rm parton} = \theta(-\omega)\,\frac{C_F\alpha_s}{\pi}\left(\frac{3}{\epsilon} - 3\ln\frac{\omega^2}{\mu^2} - 5\right). \quad (4.3)$$

Inserting (4.1) and (4.3) into (2.9), and setting $s(\omega)$, $t(\omega)$, $v(\omega)$, $f_u(\omega)$ and $f_v(\omega)$ to zero, we find:

$$W_{ij}^{(2)} = \theta(n\cdot p)\frac{C_F\alpha_s}{4\pi}\Bigg\{\mathrm{tr}\left[\Gamma_i\,\not{n}\,\Gamma_j\,\frac{1+\not{v}}{2}\right]\frac{1}{\bar{n}\cdot p}\left(\frac{3}{\epsilon} - 3\ln\frac{(n\cdot p)^2}{\mu^2} - 5\right)$$

$$+\mathrm{tr}\left[\Gamma_i\,\not{n}\,\Gamma_j\,(\not{\bar{n}} - \not{n})\right]\frac{1}{m_b}\,\frac{1}{4}\left(-\frac{1}{\epsilon} + \ln\frac{(n\cdot p)^2}{\mu^2} + 1\right)\Bigg\}, \quad (4.4)$$

which is the total contribution of the soft region, i.e. the sum of equations (3.11) and (3.12).

At this point we should note that the question of operator mixing and renormalization with regard to $u(\omega)$ is still open, as it was not considered in [49]. The main complication

JHEP06(2009)083

being the need to introduce new subleading shape functions which $u(\omega)$ can mix into, and establishing the closure of the basis. The analysis of this question goes beyond the scope of this paper. For our purposes the important point is that the terms of the form $h^0 \cdot j^0 \otimes s^1$, that were already calculated in the literature, reproduce the contribution of the soft region. As a result in the final factorization formula, no new terms are needed to account for this contribution.

4.2 Hard-Collinear contribution

In order to reproduce the result of the hard-collinear region, we need to consider various combinations of the SCET currents and the SCET Lagrangian.[5] Symbolically we need the following combinations:

$$J^{\dagger(1)}\, J^{(1)}, \quad J^{\dagger(2)}\, J^{(0)} + J^{\dagger(0)}\, J^{(2)}, \quad J^{\dagger(2)}\, J^{(0)} \int d^4z\, \mathcal{L}_\xi^{(0)} + J^{\dagger(0)}\, J^{(2)} \int d^4z\, \mathcal{L}_\xi^{(0)}, \tag{4.5}$$

where $\mathcal{L}_\xi^{(0)}$ and $J^{(i)}$ are defined in equations (2.18) and (2.21), respectively. Calculating the contribution of each combination we find that the non zero combinations are:

- $J^{\dagger(1)}\, J^{(1)}$

$$\begin{aligned} W_{ij} = \theta(p^2)\frac{C_F\alpha_s}{4\pi}\Bigg\{ &\mathrm{tr}\left[\Gamma_i\, \bar{\not{n}}\, \Gamma_j\, \frac{1+\not{v}}{2}\right]\frac{1}{\bar{n}\cdot p}\,\frac{1}{4} + \mathrm{tr}\left[\Gamma_i\, \not{n}\, \Gamma_j\, \not{n}\right]\frac{\bar{n}\cdot p}{m_b^2}\,\frac{1}{16} \\ &+ \left(\mathrm{tr}\left[\Gamma_i\, \bar{\not{n}}\, \not{n}\, \gamma_\perp^\alpha\, \Gamma_j\, \not{n}\gamma_\alpha^\perp\, \frac{1+\not{v}}{2}\right] + \mathrm{tr}\left[\Gamma_i\, \not{n}\, \bar{\not{n}}\, \gamma_\perp^\alpha\, \Gamma_j\, \frac{1+\not{v}}{2}\, \gamma_\alpha^\perp\, \not{n}\right]\right)\frac{1}{m_b}\,\frac{1}{32}\Bigg\} \end{aligned} \tag{4.6}$$

- $J^{\dagger(2)}\, J^{(0)} + J^{\dagger(0)}\, J^{(2)}, \quad J^{(2)} = -\,\bar{\mathfrak{X}}\,\Gamma\frac{\not{n}}{2m_b}\, n\cdot\mathcal{A}_{hc}\,(S^\dagger h)_{x_-}$

$$W_{ij} = \theta(p^2)\frac{C_F\alpha_s}{4\pi}\left\{\ \mathrm{tr}\left[\Gamma_i\, \not{n}\, \Gamma_j\, \not{n}\right]\frac{1}{m_b}\,\frac{1}{4}\left(-\frac{1}{\epsilon} - 1 - \ln\frac{\mu^2}{p^2}\right)\right\} \tag{4.7}$$

- $J^{\dagger(2)}\, J^{(0)} + J^{\dagger(0)}\, J^{(2)}, \quad J^{(2)} = -\,\bar{\mathfrak{X}}\,\Gamma\,\frac{1}{i\bar{n}\cdot\partial}\, n\cdot\mathcal{A}_{hc}\,(S^\dagger h)_{x_-}$

$$W_{ij} = \theta(p^2)\frac{C_F\alpha_s}{4\pi}\left\{\ \mathrm{tr}\left[\Gamma_i\, \not{n}\, \Gamma_j\, \frac{1+\not{v}}{2}\right]\frac{1}{\bar{n}\cdot p}\left(-\frac{1}{\epsilon} + 2 - \ln\frac{\mu^2}{p^2}\right)\right\} \tag{4.8}$$

- $J^{\dagger(2)}\, J^{(0)}\int d^4z\,\mathcal{L}_\xi^{(0)} + J^{\dagger(0)}\, J^{(2)}\int d^4z\,\mathcal{L}_\xi^{(0)}, \quad J^{(2)} = -\,\bar{\mathfrak{X}}\,\Gamma\frac{\not{n}}{2m_b}\, n\cdot\mathcal{A}_{hc}\,(S^\dagger h)_{x_-}$

$$W_{ij} = \theta(p^2)\frac{C_F\alpha_s}{4\pi}\left\{\ \mathrm{tr}\left[\Gamma_i\, \not{n}\, \Gamma_j\, \not{n}\right]\frac{1}{m_b}\,\frac{1}{4}\left(-\frac{1}{\epsilon} - \frac{3}{2} - \ln\frac{\mu^2}{p^2}\right)\right\} \tag{4.9}$$

[5] It is easy to show that the TOP of a first order current with the zeroth order current vanish since the perpendicular components of the hard-collinear momentum can be chosen to be zero. This corresponds of course to the absence of $1/\sqrt{m_b}$ corrections to the hadronic tensor.

JHEP06(2009)083

- $J^{\dagger(2)}\, J^{(0)} \int d^4z\, \mathcal{L}_\xi^{(0)} + J^{\dagger(0)}\, J^{(2)} \int d^4z\, \mathcal{L}_\xi^{(0)}, \quad J^{(2)} = -\,\bar{\mathfrak{X}}\,\Gamma\,\frac{1}{i\bar{n}\cdot\partial}\, n\cdot\mathcal{A}_{hc}\,(S^\dagger h)_{x_-}$

$$W_{ij} = \theta(p^2)\frac{C_F\alpha_s}{4\pi}\,\left\{\,\mathrm{tr}\left[\Gamma_i\,\not{n}\,\Gamma_j\,\frac{1+\not{v}}{2}\right]\frac{2}{\bar{n}\cdot p}\left(-\frac{1}{\epsilon}+\frac{1}{2}-\ln\frac{\mu^2}{p^2}\right)\right\} \tag{4.10}$$

- $J^{\dagger(2)}\, J^{(0)} \int d^4z\, \mathcal{L}_\xi^{(0)} + J^{\dagger(0)}\, J^{(2)} \int d^4z\, \mathcal{L}_\xi^{(0)}, \quad J^{(2)} = -\,\bar{\mathfrak{X}}\,\Gamma\,\frac{1}{i\bar{n}\cdot\partial}\,\frac{(i\not{\mathcal{D}}_{\perp hc}\,\not{\mathcal{A}}_{\perp hc})}{m_b}\,(S^\dagger h)_{x_-}$

$$\begin{aligned} W_{ij} = \theta(p^2)\frac{C_F\alpha_s}{4\pi}\Bigg\{&\mathrm{tr}\left[\Gamma_i\,\not{n}\,\Gamma_j\,\frac{1+\not{v}}{2}\right]\frac{1}{m_b}\left(-\frac{1}{\epsilon}-\frac{3}{2}-\ln\frac{\mu^2}{p^2}\right)\\ &-\left(\mathrm{tr}\left[\Gamma_i\,\not{n}\,\gamma_\perp^\beta\,\gamma_\perp^\alpha\,\Gamma_j\,\gamma_\alpha^\perp\,\gamma_\beta^\perp\,\frac{1+\not{v}}{2}\right]+\mathrm{tr}\left[\Gamma_i\,\gamma_\perp^\alpha\,\gamma_\perp^\beta\,\not{n}\,\Gamma_j\,\frac{1+\not{v}}{2}\,\gamma_\beta^\perp\,\gamma_\alpha^\perp\right]\right)\frac{1}{m_b}\,\frac{1}{16}\Bigg\} \end{aligned} \tag{4.11}$$

- $J^{\dagger(2)}\, J^{(0)} \int d^4z\, \mathcal{L}_\xi^{(0)} + J^{\dagger(0)}\, J^{(2)} \int d^4z\, \mathcal{L}_\xi^{(0)}, \quad J^{(2)} = \bar{\mathfrak{X}}\,\frac{i\overleftarrow{\not{\mathcal{D}}}_{\perp hc}}{m_b}\,\frac{1}{i\bar{n}\cdot\overleftarrow{\partial}}\,\frac{\not{\bar{n}}}{2}\,\Gamma\,\frac{\not{n}}{2}\,\not{\mathcal{A}}_{\perp hc}\,(S^\dagger h)_{x_-}$

$$\begin{aligned} W_{ij} = \theta(p^2)\frac{C_F\alpha_s}{4\pi}\Bigg\{&\mathrm{tr}\left[\Gamma_i\,\not{n}\,\not{\bar{n}}\,\gamma_\perp^\alpha\,\Gamma_j\,\not{n}\gamma_\alpha^\perp\,\frac{1+\not{v}}{2}\right]\\ &+\mathrm{tr}\left[\Gamma_i\,\not{\bar{n}}\,\not{n}\,\gamma_\perp^\alpha\,\Gamma_j\,\frac{1+\not{v}}{2}\,\gamma_\alpha^\perp\,\not{n}\right]\Bigg\}\frac{1}{m_b}\,\frac{1}{16} \end{aligned} \tag{4.12}$$

Using $\gamma_\perp^\beta = \gamma^\beta - \bar{n}^\beta\not{n}/2 - n^\beta\not{\bar{n}}/2$ and $2\not{v} = \not{n} + \not{\bar{n}}$, we can show that the sum of (4.6)–(4.12) is equal to the hard-collinear contribution i.e. the sum of (3.8), (3.9), and (3.10). Again we see that the contribution of the hard-collinear region is more complicated than that of the soft region. As we will see in the next section there are seven different jet functions that contribute at one loop, compared to only two subleading shape function that are needed to reproduce the soft region.

5 Subleading jet functions

Following the calculations of the previous sections, it is clear that in order to properly factorize all the terms in the hadronic tensor that are both α_s and $1/m_b$ suppressed in the end point region, we have keep the subleading shape functions' contribution and replace the so called "$1/m_b$ kinematical corrections" of equations (2.13) and (2.14) by the contribution of the subleading jet functions. By "jet function" we mean the discontinuity of a Fourier transform of a vacuum expectation value of a time ordered product of hard-collinear fields. By "subleading" we mean that these functions scale as $\mathcal{O}(\lambda^0)$ in the end point region.

Before going into the details of the analysis of each subleading jet function, we wish to make some general remarks. The first issue is factorization. In the following we establish

JHEP06(2009)083

factorization formula for terms of the form $h^0 \cdot j^1 \otimes s^0$. More explicitly, we always have $h^0 \equiv 1$ and s^0 is the leading order shape function. The hadronic tensor factorizes as

$$W_{ij} = \sum_a C_a \, \mathrm{tr} \left[\Gamma_i \, \ldots \, \Gamma_j \, \ldots \right] \int d\omega \, j_a(p_\omega^2) \, S(\omega). \tag{5.1}$$

Here C_a is a simple kinematical factor of mass dimension -1, e.g. $1/\bar{n} \cdot p$, $1/m_b$ and the argument of the jet function j_a is $p_\omega^2 = \bar{n} \cdot p \, (n \cdot p + \omega)$. The factorization formula can be proven in an analogous way to the leading order factorization proof as presented, for example, in [38]. The only difference is that we have subleading jet functions instead of a leading jet function.

The second issue, which does not arise at leading order, is the "correct" definition of the subleading jet functions and the role of parity and time reversal (PT) symmetry. In general the subleading jet function is the discontinuity of TOP of two *different* combinations of hard-collinear fields O_a and O_b. Here O can be a hard-collinear quark or a product of hard-collinear quark and a hard-collinear gluon, or even a more complicated object. As a result we typically have both

$$\int d^4x \, e^{-ipx} \left\langle \Omega \left| T \left\{ O_a^\dagger(0), O_b(x) \right\} \right| \Omega \right\rangle \quad \text{and} \quad \int d^4x \, e^{-ipx} \left\langle \Omega \left| T \left\{ O_b^\dagger(0), O_a(x) \right\} \right| \Omega \right\rangle . \tag{5.2}$$

The subleading jet function(s) should be the discontinuity of the *sum* of the two terms. In practice we find that after we decompose each TOP according to the different color and Lorentz structures, we can use translation invariance and the PT symmetry of the strong interaction to relate the two TOPs. We can therefore define the subleading jet function as the discontinuity of the coefficient of a specific structure in *either* TOP.

We will illustrate both issues in the more detailed calculation of the first subleading jet function. These details will be suppressed in the derivation of subsequent jet functions. As before, the one loop calculation are performed in $d = 4 - 2\epsilon$ dimensions and we implicitly take $\mu \to \mu e^{\gamma_E/2}(4\pi)^{-1/2}$.

The following list of the subleading jet functions is not necessarily the complete list of possible subleading jet functions. For example, if the QCD currents are matched onto SCET beyond tree level we would expect more complicated functions which can depend on more than one variable. We wish to emphasize, though, that the list is sufficient to describe all the terms of the form of equation (5.1) at zeroth order in $\alpha_s(\mu_h)$ and to all orders in $\alpha_s(\mu_i)$. In particular it includes all the terms of the form $h^0 \cdot j^1 \otimes s^0$ at $\mathcal{O}(\alpha_s)$.

5.1 The list of subleading jet functions

5.1.1 j_n

This jet function arises from the TOP of the leading order current $J^{(0)} = e^{-im_b v\cdot x} \, \bar{\mathfrak{X}} \, \Gamma \left(S^\dagger h \right)_{x_-}$ and the second order current $J^{(2)} = -e^{-im_b v\cdot x} \, \bar{\mathfrak{X}} \, \Gamma \frac{\not{\bar{n}}}{2m_b} \, n \cdot \mathcal{A}_{hc} \left(S^\dagger h \right)_{x_-}$. These currents are matched at tree level and as a result the Wilson coefficients equal 1. Consequently, the hard function, which is simply the product of the Wilson coefficients, equals to 1 also. The contribution of this combination of currents to the hadronic tensor is given by

$$\begin{aligned} W_{ij} = & \frac{1}{2\pi M_B} \mathrm{Im} \left\langle \bar{B} \middle| i \int d^4x \, e^{i(q-m_b v)\cdot x} \right. \\ & \times T \left\{ \left(\bar{h}S\right) \Gamma_i \, \mathfrak{X}(0), \, -\bar{\mathfrak{X}} \, \Gamma_j \frac{\not{n}}{2m_b} \, n \cdot \mathcal{A}_{hc} \left(S^\dagger h\right)_{x_-} \right\} \left| \bar{B} \right\rangle \\ & + \frac{1}{2\pi M_B} \mathrm{Im} \left\langle \bar{B} \middle| i \int d^4x \, e^{i(q-m_b v)\cdot x} \right. \\ & \times T \left\{ - \left(\bar{h}S\right) n \cdot \mathcal{A}_{hc} \frac{\not{n}}{2m_b} \Gamma_i \, \mathfrak{X}(0), \, \bar{\mathfrak{X}} \, \Gamma_j \left(S^\dagger h\right)_{x_-} \right\} \left| \bar{B} \right\rangle . \end{aligned} \tag{5.3}$$

The leading order Lagrangian does not contain any interactions between hard-collinear and soft fields. Since the B-meson states contain only soft particles, we should take the vacuum matrix element of the hard-collinear fields. From equation (5.3) we seem to have two such matrix elements, but these are related by the PT symmetry of the strong interaction, as explained in appendix A. Consequently, we have

$$\begin{aligned} \frac{\not{n}}{2} \delta_{kl} \, \mathcal{J}_n(p^2) & \stackrel{\text{def.}}{=} \int d^4x \, e^{-ip\cdot x} \langle \Omega | \, T \left\{ \mathfrak{X}_k(0), \, \left[\bar{\mathfrak{X}} \, n \cdot \mathcal{A}_{hc}\right]_l (x) \right\} | \Omega \rangle \\ & \stackrel{PT}{=} \int d^4x \, e^{-ip\cdot x} \langle \Omega | \, T \left\{ \left[n \cdot \mathcal{A}_{hc} \, \mathfrak{X}\right]_l (0), \, \bar{\mathfrak{X}}_k(x) \right\} | \Omega \rangle , \end{aligned} \tag{5.4}$$

where k, l are color indices. Inserting this definition into (5.3), we have

$$\begin{aligned} W_{ij} = & -\mathrm{Im} \int d^4x \, e^{i(q-m_b v)\cdot x} \int \frac{d^4p}{(2\pi)^4} \, e^{ip\cdot x} \, \frac{i}{\pi} \, \mathcal{J}_n(p^2) \\ & \frac{1}{m_b} \frac{1}{2M_B} \left\langle \bar{B} \middle| \left[\left(\bar{h}S\right)_0 \Gamma_i \frac{\not{n}}{2} \Gamma_j \frac{\not{n}}{2} \left(S^\dagger h\right)_{x_-} + \left(\bar{h}S\right)_0 \frac{\not{n}}{2} \Gamma_i \frac{\not{n}}{2} \Gamma_j \left(S^\dagger h\right)_{x_-} \right] \middle| \bar{B} \right\rangle . \end{aligned} \tag{5.5}$$

Defining the subleading jet function j_n as

$$j_n(p^2) = \frac{1}{\pi} \mathrm{Im} \left[i \, \mathcal{J}_n \left(p^2\right) \right] , \tag{5.6}$$

and using the definition of the leading order shape function [27, 38]

$$\frac{\langle \bar{B}(v) | \left(\bar{h}S\right)_0 \Gamma \left(S^\dagger h\right)_{x_-} | \bar{B}(v) \rangle}{2M_B} = \frac{1}{2} \, \mathrm{tr} \left(\Gamma \, \frac{1+\not{v}}{2} \right) \int d\omega \, e^{-\frac{i}{2}\omega \bar{n}\cdot x} \, S(\omega), \tag{5.7}$$

we finally find the factorization formula

$$W_{ij} = -\frac{1}{8m_b} \mathrm{tr} \left[\Gamma_i \not{n} \, \Gamma_j \not{n} \right] \int d\omega \, j_n(p_\omega^2) \, S(\omega) \, , \tag{5.8}$$

which is of the general form of equation (5.1).

An explicit one loop calculation of the bare subleading jet function j_n, which can also be extracted from the sum of equations (4.7) and (4.9), gives

$$j_n(p^2) = \theta(p^2) \frac{C_F \alpha_s}{4\pi} \cdot 4 \left(\frac{1}{\epsilon} + \frac{5}{4} + \ln \frac{\mu^2}{p^2} \right) . \tag{5.9}$$

JHEP06(2009)083

5.1.2 $j_{n'}$

This jet function arises from the TOP of the leading order current $J^{(0)} = e^{-im_b v\cdot x}\,\bar{\mathfrak{X}}\,\Gamma\,\left(S^\dagger h\right)_{x_-}$ and the second order current $J^{(2)} = \,-\,e^{-im_b v\cdot x}\,\bar{\mathfrak{X}}\,\Gamma\,\frac{1}{i\bar{n}\cdot\partial}\,n\cdot\mathcal{A}_{hc}\left(S^\dagger h\right)_{x_-}$. We define

$$\begin{aligned}\frac{1}{\bar{n}\cdot p}\,\frac{\not{n}}{2}\delta_{kl}\,\mathcal{J}_{n'}(p^2) &= \int d^4x\,e^{-ip\cdot x}\left\langle\Omega\left|T\left\{\mathfrak{X}_k(0),\,\left[\bar{\mathfrak{X}}\,\frac{1}{i\bar{n}\cdot\partial}\,n\cdot\mathcal{A}_{hc}\right]_l(x)\right\}\right|\Omega\right\rangle\\ &= \int d^4x\,e^{-ip\cdot x}\left\langle\Omega\left|T\left\{\left[n\cdot\mathcal{A}_{hc}\,\frac{1}{-i\bar{n}\cdot\overleftarrow{\partial}}\,\mathfrak{X}\right]_l(0),\,\bar{\mathfrak{X}}_k(x)\right\}\right|\Omega\right\rangle,\end{aligned} \tag{5.10}$$

where k,l are color indices. We define the subleading jet function $j_{n'}$ as,

$$j_{n'}(p^2) = \frac{1}{\pi}\mathrm{Im}\left[i\,\mathcal{J}_{n'}(p^2)\right]. \tag{5.11}$$

The contribution of $j_{n'}$ to the hadronic tensor is

$$W_{ij} = -\frac{1}{\bar{n}\cdot p}\,\mathrm{tr}\left[\Gamma_i\,\frac{\not{n}}{2}\,\Gamma_j\,\frac{1+\not{v}}{2}\right]\int d\omega\,j_{n'}(p^2_\omega)\,S(\omega). \tag{5.12}$$

An explicit one loop calculation of the bare subleading jet function $j_{n'}$, which can also be extracted from the sum of equations (4.8) and (4.10), gives

$$j_{n'}(p^2) = \theta(p^2)\frac{C_F\alpha_s}{4\pi}\cdot 6\left(\frac{1}{\epsilon} - 1 + \ln\frac{\mu^2}{p^2}\right). \tag{5.13}$$

5.1.3 j^S_{11} and j^A_{11}

These two jet functions arise when combining two first order currents. Recall from (2.21) that the first order current is

$$J^{(1)} = \,-\,e^{-im_b v\cdot x}\,\bar{\mathfrak{X}}\,\frac{\not{\bar{n}}}{2}\,\not{\mathcal{A}}_{\perp hc}\,\frac{1}{i\bar{n}\cdot\overleftarrow{\partial}}\,\Gamma\left(S^\dagger h\right)_{x_-} - e^{-im_b v\cdot x}\,\bar{\mathfrak{X}}\,\Gamma\,\frac{\not{n}}{2m_b}\,\not{\mathcal{A}}_{\perp hc}\left(S^\dagger h\right)_{x_-}. \tag{5.14}$$

Since there are two terms in the first order current, one might naively assume that we will need to define a different jet function for each pair of terms. This is not the case, as we explain in detail in appendix B. Schematically, the reason is that the inverse derivative acts on *all* the hard-collinear fields in the first term of (5.14), which translates into an overall $1/\bar{n}\cdot p$ factor.

For all the terms we need to consider the following decomposition of the TOP of hard-collinear fields,

$$\begin{aligned}&\int d^4x\,e^{-ip\cdot x}\left\langle\Omega\left|T\Big\{\left[\mathcal{A}^\mu_{\perp hc}\,\mathfrak{X}\right]_k(0)\,,\left[\bar{\mathfrak{X}}\,\mathcal{A}^\nu_{\perp\,hc}\right]_l(x)\Big\}\right|\Omega\right\rangle =\\ &\qquad\qquad \bar{n}\cdot p\,\frac{\not{n}}{2}\,\frac{g^{\mu\nu}_\perp}{d-2}\,\delta_{kl}\,\mathcal{J}^S_{11}(p^2) + \bar{n}\cdot p\,\frac{\not{n}}{2}\,\gamma_5\,\frac{i\epsilon^{\mu\nu}_\perp}{d-2}\,\delta_{kl}\,\mathcal{J}^A_{11}(p^2).\end{aligned} \tag{5.15}$$

It is clear that $g^{\mu\nu}_\perp$ and $i\epsilon^{\mu\nu}_\perp$ are the only possible tensors the TOP can depend on. The Dirac structure that accompanies each tensor is determined by PT invariance (see appendix A). We now define as usual,

$$j^S_{11}(p^2) = \frac{1}{\pi}\mathrm{Im}\left[i\,\mathcal{J}^S_{11}(p^2)\right] \quad\text{and}\quad j^A_{11}(p^2) = \frac{1}{\pi}\mathrm{Im}\left[i\,\mathcal{J}^A_{11}(p^2)\right]. \tag{5.16}$$

JHEP06(2009)083

By an explicit calculation one can show that while j_{11}^S starts $\mathcal{O}(\alpha_s)$, j_{11}^A is non-zero only at $\mathcal{O}(\alpha_s^2)$. The one loop result for j_{11}^S is

$$j_{11}^S(p^2) = -\theta(p^2)\frac{C_F\alpha_s}{4\pi}. \tag{5.17}$$

The contribution of j_{11}^S and j_{11}^A to the hadronic tensor is

$$\begin{aligned} W_{ij} = & -\frac{\bar{n}\cdot p}{16m_b^2}\,\mathrm{tr}\,[\Gamma_i\,\not{n}\,\Gamma_j\,\not{n}]\int d\omega\; j_{11}^S(p_\omega^2)\,S(\omega) - \frac{\bar{n}\cdot p}{16m_b^2}\,\mathrm{tr}\,[\Gamma_i\,\not{n}\gamma_5\,\Gamma_j\,\not{n}\gamma_5]\int d\omega\; j_{11}^A(p_\omega^2)\,S(\omega) \\ & -\left\{\frac{1}{4\bar{n}\cdot p}\,\mathrm{tr}\left[\Gamma_i\,\not{\bar{n}}\,\Gamma_j\,\frac{1+\not{v}}{2}\right] - \frac{1}{16m_b}\,\mathrm{tr}\left[\Gamma_i\,\gamma_\rho^\perp\,\Gamma_j\,\gamma_\perp^\rho\right] - \frac{1}{16m_b}\,\mathrm{tr}\left[\Gamma_i\,\gamma_\rho^\perp\gamma_5\,\Gamma_j\,\gamma_\perp^\rho\gamma_5\right]\right\} \\ & \times\int d\omega\,\left[j_{11}^S(p_\omega^2) + j_{11}^A(p_\omega^2)\right]\,S(\omega). \end{aligned} \tag{5.18}$$

5.1.4 j_G and j_K

This jet function arises from the TOP of the leading order current $J^{(0)} = e^{-im_b v\cdot x}\,\bar{\mathfrak{X}}\,\Gamma\,\left(S^\dagger h\right)_{x_-}$ and the second order current $J^{(2)} = -\,e^{-im_b v\cdot x}\,\bar{\mathfrak{X}}\,\Gamma\,\frac{1}{i\bar{n}\cdot\partial}\,\frac{(i\not{\mathcal{D}}_{\perp hc}\,\not{\mathcal{A}}_{\perp hc})}{m_b}\,\left(S^\dagger h\right)_{x_-}$. We find it more useful to use the original form of the current as it appeared in [14, 50]. Using the identity

$$i\not{D}_\perp\, i\not{D}_\perp = (iD_\perp)^2 + \frac{g}{2}\,\sigma_{\mu\nu}^\perp G_\perp^{\mu\nu}, \tag{5.19}$$

we find that the current can be written as,

$$\begin{aligned} \bar{\mathfrak{X}}\,\Gamma\,\frac{1}{i\bar{n}\cdot\partial}\,\frac{(i\not{\mathcal{D}}_{\perp hc}\,\not{\mathcal{A}}_{\perp hc})}{m_b}\left(S^\dagger h\right)_{x_-} &= \bar{\xi}\,\Gamma\,\frac{1}{i\bar{n}\cdot D}\,\frac{\left[i\not{D}_{\perp hc}\,i\not{D}_{\perp hc}\,W\right]}{m_b}\left(S^\dagger h\right)_{x_-} \\ &= \bar{\mathfrak{X}}\,\Gamma\,\frac{1}{i\bar{n}\cdot\partial}\,\frac{\left[W^\dagger\,(iD_{\perp hc})^2\,W + \frac{g}{2}W^\dagger\,\sigma_{\mu\nu}G^{\mu\nu}\,W\right]}{m_b}\left(S^\dagger h\right)_{x_-}. \end{aligned} \tag{5.20}$$

Since $(iD_{\perp hc})^2$ is even under PT symmetry, while $G^{\mu\nu}$ is odd, we need to separate the two parts of this current. We define,

$$\begin{aligned} \frac{\not{n}}{2}\delta_{kl}\,\mathcal{J}_K(p^2) &= \int d^4x\,e^{-ip\cdot x}\left\langle\Omega\left|T\left\{\mathfrak{X}_k(0),\,\left[\bar{\mathfrak{X}}\,\frac{1}{i\bar{n}\cdot\partial}\,W^\dagger\,(iD_{\perp hc})^2\,W\right]_l(x)\right\}\right|\Omega\right\rangle \\ &= \int d^4x\,e^{-ip\cdot x}\left\langle\Omega\left|T\left\{\left[W^\dagger\,\left(i\overleftarrow{D}_{\perp hc}\right)^2\,W\,\frac{1}{-i\bar{n}\cdot\overleftarrow{\partial}}\,\mathfrak{X}\right]_l(0),\,\bar{\mathfrak{X}}_k(x)\right\}\right|\Omega\right\rangle, \end{aligned} \tag{5.21}$$

$$\begin{aligned} \frac{i\epsilon_\perp^{\mu\nu}}{d-2}\,\frac{\not{n}}{2}\gamma_5\,\delta_{kl}\,\mathcal{J}_G(p^2) &= \int d^4x\,e^{-ip\cdot x}\left\langle\Omega\left|T\left\{\mathfrak{X}_k(0),\,\left[\bar{\mathfrak{X}}\,\frac{1}{i\bar{n}\cdot\partial}\,W^\dagger\,G^{\mu\nu}\,W\right]_l(x)\right\}\right|\Omega\right\rangle \\ &= \int d^4x\,e^{-ip\cdot x}\left\langle\Omega\left|T\left\{\left[W^\dagger\,G^{\mu\nu}\,W\,\frac{1}{-i\bar{n}\cdot\overleftarrow{\partial}}\,\mathfrak{X}\right]_l(0),\,\bar{\mathfrak{X}}_k(x)\right\}\right|\Omega\right\rangle, \end{aligned} \tag{5.22}$$

and the corresponding subleading jet functions,

$$j_K(p^2) = \frac{1}{\pi}\mathrm{Im}\left[i\,\mathcal{J}_K(p^2)\right] \quad \text{and} \quad j_G(p^2) = \frac{1}{\pi}\mathrm{Im}\left[i\,\mathcal{J}_G(p^2)\right]. \tag{5.23}$$

Their contribution to the hadronic tensor is

$$\begin{aligned} W_{ij} = &-\frac{1}{m_b}\,\mathrm{tr}\left[\Gamma_i\,\frac{\not{n}}{2}\,\Gamma_j\,\frac{1+\not{v}}{2}\right]\int d\omega\, j_K(p^2_\omega)\,S(\omega) \\ &-\frac{1}{16m_b}\,\mathrm{tr}\left[\,\Gamma_i\,\not{n}\gamma_5\,\Gamma_j\,\left(\not{\bar{n}}-\not{n}\right)\gamma_5\right]\int d\omega\, j_G(p^2_\omega)\,S(\omega). \end{aligned} \tag{5.24}$$

An explicit one loop calculation gives,

$$\begin{aligned} j_K(p^2) &= \theta(p^2)\frac{C_F\alpha_s}{4\pi}\cdot(-2)\left(\frac{1}{\epsilon}+\frac{5}{4}+\ln\frac{\mu^2}{p^2}\right) \\ j_G(p^2) &= -\theta(p^2)\frac{C_F\alpha_s}{4\pi}. \end{aligned} \tag{5.25}$$

Notice the interesting fact that at one loop $2j_K+j_n=0$. Similarly $2\mathcal{J}_K+\mathcal{J}_n=0$, which is true even without expanding in ϵ. It is unclear whether this is a one loop "accident" or a more general result that holds to all orders in perturbation theory.

5.1.5 j_A and j_S

This jet function arises from the TOP of the leading order current $J^{(0)} = e^{-im_bv\cdot x}\,\bar{\mathfrak{X}}\,\Gamma\left(S^\dagger h\right)_{x_-}$ and the second order current $J^{(2)} = -\,e^{-im_bv\cdot x}\,\bar{\mathfrak{X}}\,\frac{i\overleftarrow{\not{\mathcal{D}}}_{\perp hc}}{m_b}\times \times\frac{1}{i\bar{n}\cdot\overleftarrow{\partial}}\,\frac{\not{\bar{n}}}{2}\,\Gamma\,\frac{\not{n}}{2}\,\not{\mathcal{A}}_{\perp hc}\left(S^\dagger h\right)_{x_-}$. We define

$$\begin{aligned} \int d^4x\, e^{-ip\cdot x}\left\langle\Omega\middle|\,T\Big\{\mathfrak{X}_k(0),\left[\bar{\mathfrak{X}}\,i\overleftarrow{\mathcal{D}}^{\mu}_{\perp hc}\frac{1}{i\bar{n}\cdot\overleftarrow{\partial}}\,\mathcal{A}^{\nu}_{\perp\,hc}\right]_l(x)\Big\}\middle|\,\Omega\right\rangle = \\ \frac{\not{n}}{2}\frac{g_\perp^{\mu\nu}}{d-2}\,\delta_{kl}\,\mathcal{J}_S\left(p^2\right) + \frac{\not{n}}{2}\,\gamma_5\frac{i\epsilon_\perp^{\mu\nu}}{d-2}\,\delta_{kl}\,\mathcal{J}_A\left(p^2\right). \end{aligned} \tag{5.26}$$

PT symmetry implies that

$$\begin{aligned} \int d^4x\, e^{-ip\cdot x}\left\langle\Omega\middle|\,T\Big\{\left[\mathcal{A}^{\nu}_{\perp\,hc}\,\frac{1}{i\bar{n}\cdot\partial}\,i\mathcal{D}^{\mu}_{\perp hc}\,\mathfrak{X}\right]_l(0),\bar{\mathfrak{X}}_k(x)\Big\}\middle|\,\Omega\right\rangle = \\ \frac{\not{n}}{2}\frac{g_\perp^{\mu\nu}}{d-2}\,\delta_{kl}\,\mathcal{J}_S(p^2) - \frac{\not{n}}{2}\,\gamma_5\frac{i\epsilon_\perp^{\mu\nu}}{d-2}\,\delta_{kl}\,\mathcal{J}_A(p^2). \end{aligned} \tag{5.27}$$

Notice the minus sign in front of the second term.

We now define the corresponding subleading jet functions,

$$j_S\left(p^2\right) = \frac{1}{\pi}\mathrm{Im}\left[i\,\mathcal{J}_S\left(p^2\right)\right] \quad \text{and} \quad j_A\left(p^2\right) = \frac{1}{\pi}\mathrm{Im}\left[i\,\mathcal{J}_A\left(p^2\right)\right]. \tag{5.28}$$

Their contribution to the hadronic tensor is

$$
\begin{aligned}
W_{ij} \,=\, & \frac{1}{16m_b}\left\{ \mathrm{tr}\left[\Gamma_i\,\gamma^\perp_\rho\,\Gamma_j\,\gamma^\rho_\perp\right] - \mathrm{tr}\left[\Gamma_i\,\gamma^\perp_\rho\gamma_5\,\Gamma_j\,\gamma^\rho_\perp\gamma_5\right]\right\} \\
& \times\int d\omega\,\left[j_S(p^2_\omega)+(3-d)\,j_A(p^2_\omega)\right]\,S(\omega).
\end{aligned} \tag{5.29}
$$

An explicit one loop calculation gives,

$$
\begin{aligned}
j_S(p^2) \,&=\, \theta(p^2)\frac{C_F\alpha_s}{4\pi}\cdot\left(-\frac{3}{2}\right) \\
j_A(p^2) \,&=\, \theta(p^2)\frac{C_F\alpha_s}{4\pi}\cdot\left(\frac{1}{2}\right).
\end{aligned} \tag{5.30}
$$

5.2 Renormalization

Having defined and calculated all the subleading jet function to one loop order, we are ready to discuss their renormalization. We note first that only 3 out of the 8 functions we have defined require renormalization at one loop order. These are

$$
\begin{aligned}
j^{\mathrm{bare}}_n(p^2,\mu) \,&=\, \theta(p^2)\frac{C_F\alpha_s}{4\pi}\cdot 4\,\left(\frac{1}{\epsilon}+\frac{5}{4}+\ln\frac{\mu^2}{p^2}\right)+\mathcal{O}(\alpha_s^2) \\
j^{\mathrm{bare}}_{n'}(p^2,\mu) \,&=\, \theta(p^2)\frac{C_F\alpha_s}{4\pi}\cdot 6\,\left(\frac{1}{\epsilon}-1+\ln\frac{\mu^2}{p^2}\right)+\mathcal{O}(\alpha_s^2) \\
j^{\mathrm{bare}}_K(p^2,\mu) \,&=\, \theta(p^2)\frac{C_F\alpha_s}{4\pi}\cdot(-2)\,\left(\frac{1}{\epsilon}+\frac{5}{4}+\ln\frac{\mu^2}{p^2}\right)+\mathcal{O}(\alpha_s^2).
\end{aligned} \tag{5.31}
$$

In order to renormalize these function we have to introduce a *new* function

$$
j_0(p^2)=\theta(p^2)+\mathcal{O}(\alpha_s), \tag{5.32}
$$

where only the *zeroth* order in α_s part of $j_0(p^2)$ is needed in order to renormalize the subleading jet functions at *first* order in α_s.

It is very tempting to relate this function to the integral over the leading order jet function, namely to identify $j_0(p^2)$ with

$$
\int^{p^2} dp'^2\,J(p'^2,\mu) \tag{5.33}
$$

(where the lower limit of the integral can be any negative number), since both are equal at $\mathcal{O}(\alpha_s^0)$. A very similar function, $j(\ln\frac{p^2}{\mu^2},\mu)$ was defined in [25]

$$
j\left(\ln\frac{p^2}{\mu^2},\mu\right)=\int_0^{p^2} dp'^2\,J(p'^2,\mu). \tag{5.34}
$$

In that paper the authors derived the two loop expression for j and its anomalous dimension. Since we only use the $\mathcal{O}(\alpha_s^0)$ expression for j_0, we will refrain from identifying it with equation (5.33) or with j of [25]. It seems plausible that all these expression are the same,

JHEP06(2009)083

but in order to determine whether they coincide beyond $\mathcal{O}(\alpha_s^0)$, a two loop calculation of the subleading jet function is needed. Such a calculation is beyond the scope of this paper.

The most convenient scheme in which to renormalize the subleading jet functions (as well as the subleading shape functions) is the $\overline{\rm DR}$ subtraction scheme [51, 52]. In this scheme the Dirac algebra is performed in $d = 4$ dimensions, while loop integrals are evaluated in $d = 4 - 2\epsilon$ dimensions. Choosing this scheme ensures that the renormalized subleading functions are the same for $\bar{B} \to X_u\, l\, \bar{\nu}$ and the $Q_{7\gamma} - Q_{7\gamma}$ contribution to $\bar{B} \to X_s\, \gamma$.

We renormalize the subleading jet function in the following way. Define the matrix $Z(p^2, p'^2, \mu)$ via

$$\begin{pmatrix} j_n(p^2) \\ j_{n'}(p^2) \\ j_K(p^2) \end{pmatrix} = \int dp'^2\, Z(p^2, p'^2, \mu) \begin{pmatrix} j_0^{\rm bare}(p'^2) \\ j_n^{\rm bare}(p'^2) \\ j_{n'}^{\rm bare}(p'^2) \\ j_K^{\rm bare}(p'^2) \end{pmatrix} . \tag{5.35}$$

At one loop order we find that in the $\overline{\rm DR}$ scheme

$$Z(p^2, p'^2, \mu) = \delta(p^2 - p'^2) \begin{pmatrix} -4a & 1 & 0 & 0 \\ -6a & 0 & 1 & 0 \\ 2a & 0 & 0 & 1 \end{pmatrix} , \tag{5.36}$$

where $a = C_F \alpha_s / 4\pi\epsilon$. Notice that $Z(p^2, p'^2, \mu)$ is not a square matrix, since we do not renormalize j_0. If indeed j_0 is related to the integral over the leading order jet function, then Z will have to include other distributions apart from a delta function, for details, see [25].

From $Z(p^2, p'^2, \mu)$ we can find the renormalization group equations for the subleading jet functions. We have at one loop

$$\frac{d}{d\ln\mu} \begin{pmatrix} j_n(p^2) \\ j_{n'}(p^2) \\ j_K(p^2) \end{pmatrix} = \int dp'^2\, \delta(p^2 - p'^2)\, \frac{C_F \alpha_s}{4\pi} \begin{pmatrix} 8 & 0 & 0 & 0 \\ 12 & 0 & 0 & 0 \\ -4 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} j_0(p'^2) \\ j_n(p'^2) \\ j_{n'}(p'^2) \\ j_K(p'^2) \end{pmatrix} , \tag{5.37}$$

The expression for the renormalized subleading jet functions are

$$\begin{aligned} j_n(p^2, \mu) &= \theta(p^2) \frac{C_F \alpha_s}{4\pi} \cdot 4 \left(\frac{5}{4} + \ln \frac{\mu^2}{p^2} \right) + \mathcal{O}(\alpha_s^2) \\ j_{n'}(p^2, \mu) &= \theta(p^2) \frac{C_F \alpha_s}{4\pi} \cdot 6 \left(-1 + \ln \frac{\mu^2}{p^2} \right) + \mathcal{O}(\alpha_s^2) \\ j_K(p^2, \mu) &= \theta(p^2) \frac{C_F \alpha_s}{4\pi} \cdot (-2) \left(\frac{5}{4} + \ln \frac{\mu^2}{p^2} \right) + \mathcal{O}(\alpha_s^2). \end{aligned} \tag{5.38}$$

It is unclear whether with the inclusion of j_0 the list of the subleading jet functions closes under renormalization. In particular one might expect that we need to define a more general subleading jet function that depends on more than one variable. For example the discontinuity of the Fourier transform of

$$\left\langle \Omega \middle| T \left\{ \mathfrak{X}_k(0),\, n\cdot \mathcal{A}^{hc}_{mn}(y),\, \bar{\mathfrak{X}}_l(x) \right\} \middle| \Omega \right\rangle \tag{5.39}$$

where k,l,m,n are color indices. An interesting fact about this type of subleading jet function is that at order g^2 it only becomes singular in the $y\to 0$ and $y\to x$ limits, where it can be related to j_n.

Clearly this topic deserves further study, but for our purposes the renormalization via the introduction of j_0 is sufficient at the order in which we are working, namely, $\alpha_s(\mu_i)/m_b$.

For phenomenological applications it is convenient to set the scale of the subleading jet functions to be the same as the scale of the leading order jet function. This is also the convenient scale in which to extract the leading order shape function [29]. At any case the resummation of the subleading logs is expected to be a small effect. As a result, the renormalization group equations of the subleading jet function are not expected to be important for phenomenological applications.

5.3 Results

We are ready to summarize our results. The subleading jet functions' contribution to the hadronic tensor can be written as

$$\begin{aligned} W^{\rm SJF}_{ij} = -\int d\omega \Bigg[& j_n(p^2_\omega,\mu)\frac{2\tilde{T}_2}{m_b} + j_{n'}(p^2_\omega,\mu)\frac{\tilde{T}_1}{\bar{n}\cdot p} + j_K(p^2_\omega,\mu)\frac{\tilde{T}_1}{m_b} + j_G(p^2_\omega,\mu)\frac{\tilde{T}_4}{m_b} \\ & + j^S_{11}(p^2_\omega,\mu)\left(\frac{\bar{n}\cdot p}{m_b^2}\tilde{T}_2 + \frac{\tilde{T}_3}{\bar{n}\cdot p} - \frac{\tilde{T}_5}{m_b} - \frac{\tilde{T}_6}{m_b}\right) \\ & + j^A_{11}(p^2_\omega,\mu)\left(\frac{\bar{n}\cdot p}{m_b^2}\tilde{T}_7 + \frac{\tilde{T}_3}{\bar{n}\cdot p} - \frac{\tilde{T}_5}{m_b} - \frac{\tilde{T}_6}{m_b}\right) \\ & + j_S(p^2_\omega,\mu)\left(\frac{\tilde{T}_5}{m_b} - \frac{\tilde{T}_6}{m_b}\right) + j_A(p^2_\omega,\mu)\left(-\frac{\tilde{T}_5}{m_b} + \frac{\tilde{T}_6}{m_b}\right)\Bigg] S(\omega), \end{aligned} \tag{5.40}$$

where the traces $\tilde{T}_1 \dots \tilde{T}_6$ are[6]

$$\begin{aligned} \tilde{T}_1 &= \frac{1}{2}\,\mathrm{tr}\left[\Gamma_i\, \not{\bar{n}}\, \Gamma_j\, \frac{1+\not{v}}{2}\right], & \tilde{T}_2 &= \frac{1}{16}\,\mathrm{tr}\left[\Gamma_i\, \not{\bar{n}}\, \Gamma_j\, \not{n}\right], \\ \tilde{T}_3 &= \frac{1}{4}\,\mathrm{tr}\left[\Gamma_i\, \not{\bar{n}}\, \Gamma_j\, \frac{1+\not{v}}{2}\right], & \tilde{T}_4 &= \frac{1}{16}\,\mathrm{tr}\left[\Gamma_i\, \not{n}\gamma_5\, \Gamma_j\, (\not{\bar{n}} - \not{n})\,\gamma_5\right], \\ \tilde{T}_5 &= \frac{1}{16}\,\mathrm{tr}\left[\Gamma_i\, \gamma^\perp_\rho\, \Gamma_j\, \gamma^\rho_\perp\right], & \tilde{T}_6 &= \frac{1}{16}\,\mathrm{tr}\left[\Gamma_i\, \gamma^\perp_\rho\gamma_5\, \Gamma_j\, \gamma^\rho_\perp\gamma_5\right] \\ \tilde{T}_7 &= \frac{1}{16}\,\mathrm{tr}\left[\Gamma_i\, \not{n}\gamma_5\, \Gamma_j\, \not{n}\gamma_5\right]. \end{aligned} \tag{5.41}$$

[6] Under the assumption that Γ_i and Γ_j contain the same number of Dirac's gamma matrices we can relate $\tilde{T}_1 \dots \tilde{T}_6$ to $T_1 \dots T_4$ in equation (2.10). The relations are $T_1 = \tilde{T}_1/2$, $T_2 = \tilde{T}_1/2 - 2\tilde{T}_2$, $T_3 = 2\tilde{T}_6$, $T_4 = \tilde{T}_4$.

The subleading jet functions $j_{11}^S, j_{11}^A, j_n, j_{n'}, j_K, j_G, j_S, j_A$ are defined in section 5.1. The renormalized one loop expressions for them are

$$\begin{aligned}
j_{11}^S(p^2,\mu) &= \theta(p^2)\frac{C_F\alpha_s(\mu)}{4\pi}\left(-1\right) + \mathcal{O}(\alpha_s^2) \\
j_{11}^A(p^2,\mu) &= 0 + \mathcal{O}(\alpha_s^2) \\
j_n(p^2,\mu) &= \theta(p^2)\frac{C_F\alpha_s(\mu)}{4\pi}\left(5 + 4\ln\frac{\mu^2}{p^2}\right) + \mathcal{O}(\alpha_s^2) \\
j_{n'}(p^2,\mu) &= \theta(p^2)\frac{C_F\alpha_s(\mu)}{4\pi}\left(-6 + 6\ln\frac{\mu^2}{p^2}\right) + \mathcal{O}(\alpha_s^2) \\
j_K(p^2,\mu) &= \theta(p^2)\frac{C_F\alpha_s(\mu)}{4\pi}\left(-\frac{5}{2} - 2\ln\frac{\mu^2}{p^2}\right) + \mathcal{O}(\alpha_s^2) \\
j_G(p^2,\mu) &= \theta(p^2)\frac{C_F\alpha_s(\mu)}{4\pi}\left(-1\right) + \mathcal{O}(\alpha_s^2) \\
j_S(p^2,\mu) &= \theta(p^2)\frac{C_F\alpha_s(\mu)}{4\pi}\left(-\frac{3}{2}\right) + \mathcal{O}(\alpha_s^2) \\
j_A(p^2,\mu) &= \theta(p^2)\frac{C_F\alpha_s(\mu)}{4\pi}\left(\frac{1}{2}\right) + \mathcal{O}(\alpha_s^2).
\end{aligned} \tag{5.42}$$

These expressions are in the $\overline{\mathrm{DR}}$ subtraction scheme [51, 52]. As explained in section 5.2 this is the most appropriate renormalization scheme for the subleading jet and shape functions.

The scale dependence in equation (5.40) cancels against the scale dependence of the subleading shape functions' contribution [27]

$$\begin{aligned}
W_{ij}^{\mathrm{SSF}} = \int d\omega\, \delta(n\cdot p+\omega)\Bigg[&\frac{\omega\, S(\omega,\mu)+t(\omega,\mu)}{m_b}T_2 + \frac{s(\omega,\mu)}{m_b}T_1 + \frac{t(\omega,\mu)}{\bar n\cdot p}T_3 + \frac{u(\omega,\mu)}{\bar n\cdot p}T_1 \\
&- \frac{v(\omega,\mu)}{\bar n\cdot p}T_4 - \pi\alpha_s\left(\frac{f_u(\omega,\mu)}{\bar n\cdot p}T_1 + \frac{f_v(\omega,\mu)}{\bar n\cdot p}T_4\right)\Bigg] + \mathcal{O}(\alpha_s).
\end{aligned} \tag{5.43}$$

For the definition of the subleading shape functions and the traces $T_1 \ldots T_4$, see section 2.

We now specialize to the cases of semileptonic and radiative B decays, using the expressions for Γ_i and Γ_j given in section 2. For the $Q_{7\gamma} - Q_{7\gamma}$ contribution to $\bar B \to X_s\gamma$ we need the "trace" of the hadronic tensor: W. Its relation to the photon spectrum of $\bar B \to X_s\gamma$ is given in equation (2.5). The subleading jet functions' contribution to W is

$$\begin{aligned}
W^{\mathrm{SJF}} = \int d\omega\Bigg[&\frac{1}{m_b}\Big(4j_K(p_\omega^2,\mu) + 4j_n(p_\omega^2,\mu) + 2j_G(p_\omega^2,\mu)\Big) + \frac{4}{\bar n\cdot p}j_{n'}(p_\omega^2,\mu) \\
&+\frac{2\bar n\cdot p}{m_b^2}\Big(j_{11}^S(p_\omega^2,\mu) - j_{11}^A(p_\omega^2,\mu)\Big)\Bigg]S(\omega).
\end{aligned} \tag{5.44}$$

Note that j_S and j_A do not contribute to W. At the lowest order in α_s this expression is

$$W^{\mathrm{SJF}} = \int d\omega\,\frac{1}{m_b}\frac{C_F\alpha_s(\mu)}{4\pi}\,\theta(p_\omega^2)\left[32\,\ln\frac{\mu^2}{p_\omega^2} - 18\right]S(\omega) + \mathcal{O}(\alpha_s^2), \tag{5.45}$$

where in order to simplify the expression, we have used the fact that for this decay mode $\bar{n}\cdot p = m_b$. For completeness we list also the subleading shape functions' contribution

$$W^{\,\rm SSF} = -\frac{2}{m_b}\int d\omega\,\delta(n\cdot p+\omega)\Big[-\;\omega\,S(\omega,\mu)+s(\omega,\mu)-t(\omega,\mu)+u(\omega,\mu)-v(\omega,\mu)$$
$$-\;\pi\alpha_s\,f_u(\omega,\mu)-\pi\alpha_s\,f_v(\omega,\mu)\Big]+\mathcal{O}(\alpha_s) \qquad (5.46)$$

For $\bar{B}\to X_u\,l\,\bar{\nu}$ we need the three "form factors": $\tilde{W}_1,\tilde{W}_2$ and $\tilde{W}_{\rm comb}\equiv\frac{y}{4}\tilde{W}_3+\tilde{W}_4+\frac{1}{y}\tilde{W}_5$. $\tilde{W}_i$ are defined in equation (2.6) and their relation to the triple differential decay rate is given by equation (2.7). The subleading jet functions' contribution is

$$\begin{aligned}
\tilde{W}_1^{\rm SJF} &= -\int d\omega\Bigg[\frac{1}{\bar{n}\cdot p}\Big(2j_{n'}(p_\omega^2,\mu)+j_{11}^S(p_\omega^2,\mu)+j_{11}^A(p_\omega^2,\mu)\Big)\\
&\qquad+\frac{1}{m_b}\Big(2j_K(p_\omega^2,\mu)+j_G(p_\omega^2,\mu)\Big)\Bigg]\,S(\omega)\\
\tilde{W}_2^{\rm SJF} &= -\int d\omega\,\frac{2}{\bar{n}\cdot p}\Big[j_{11}^S(p_\omega^2,\mu)+j_{11}^A(p_\omega^2,\mu)\Big]\,S(\omega)\\
\tilde{W}_{\rm comb}^{\rm SJF} &= -\int d\omega\Bigg[\left(\frac{4}{m_b}-\frac{2}{\bar{n}\cdot p}\right)\Big(j_{11}^S(p_\omega^2,\mu)+j_{11}^A(p_\omega^2,\mu)\Big)\\
&\qquad+\frac{2}{\bar{n}\cdot p}\Big(j_n(p_\omega^2,\mu)-j_G(p_\omega^2,\mu)\Big)\Bigg]\,S(\omega).
\end{aligned} \qquad (5.47)$$

At the order in which we are working in, it can approximated by $\bar{n}\cdot p/m_b$. As for W, j_S and j_A do not contribute to $\tilde{W}_1$, $\tilde{W}_2$ or $\tilde{W}_{\rm comb}$. At the lowest order in α_s the last equation is

$$\begin{aligned}
\tilde{W}_1^{\rm SJF} &= -\int d\omega\frac{C_F\alpha_s(\mu)}{4\pi}\,\theta(p_\omega^2)\left[\frac{1}{\bar{n}\cdot p}\left(12\ln\frac{\mu^2}{p_\omega^2}-11\right)-\frac{1}{m_b}\left(4\ln\frac{\mu^2}{p_\omega^2}+6\right)\right]S(\omega)\\
\tilde{W}_2^{\rm SJF} &= \int d\omega\,\frac{C_F\alpha_s(\mu)}{4\pi}\,\theta(p_\omega^2)\,\frac{2}{\bar{n}\cdot p}\,S(\omega)\\
\tilde{W}_{\rm comb}^{\rm SJF} &= -\int d\omega\frac{C_F\alpha_s(\mu)}{4\pi}\,\theta(p_\omega^2)\left[\frac{1}{\bar{n}\cdot p}\left(8\ln\frac{\mu^2}{p_\omega^2}+14\right)-\frac{4}{m_b}\right]S(\omega).
\end{aligned} \qquad (5.48)$$

If we are using the so called "BLNP" approach [29], i.e. using the definition of y as in equation (2.8), then some of the subleading terms are absorbed into the leading order formula (2.15). The subleading jet functions' contribution to $\tilde{W}_1$ in this case is given by

$$\tilde{W}_{1,\rm BLNP}^{\rm SJF} = -\int d\omega\frac{C_F\alpha_s(\mu)}{4\pi}\,\theta(p_\omega^2)\left[\frac{1}{\bar{n}\cdot p}\left(12\ln\frac{\mu^2}{p_\omega^2}-15\right)-\frac{1}{m_b}\left(4\ln\frac{\mu^2}{p_\omega^2}+2\right)\right]S(\omega), \qquad (5.49)$$

where there is no change to $\tilde{W}_2$ and $\tilde{W}_{\rm comb}$. For completeness we list also the subleading shape functions' contribution

$$\begin{aligned}
\tilde{W}_1^{\rm SSF} &= \int d\omega\,\delta(n\cdot p+\omega)\left[\frac{\omega\,S(\omega,\mu)+s(\omega,\mu)+t(\omega,\mu)}{m_b}+\frac{u(\omega,\mu)-v(\omega,\mu)}{\bar{n}\cdot p}\right]\\
\tilde{W}_2^{\rm SSF} &= 0\\
\tilde{W}_{\rm comb}^{\rm SSF} &= -2\int d\omega\,\delta(n\cdot p+\omega)\left[\frac{\omega\,S(\omega,\mu)+2t(\omega,\mu)}{\bar{n}\cdot p}-\frac{t(\omega,\mu)+v(\omega,\mu)}{y\,\bar{n}\cdot p}\right].
\end{aligned} \qquad (5.50)$$

6 Summary and conclusions

Decay rates of inclusive B decays, namely $\bar{B} \to X_u\, l\, \bar{\nu}$ and the $Q_{7\gamma} - Q_{7\gamma}$ contribution to $\bar{B} \to X_s\, \gamma$, are known to factorize in the end point region at the leading order in $\Lambda_{\rm QCD}/m_b$ into a product of a hard function and a universal leading order jet function convoluted with a universal leading order shape function. The hard and jet function are calculable in perturbation theory while the shape function is a non perturbative object. Recently the hard function for semileptonic decays was calculated at $\mathcal{O}(\alpha_s^2)$ [15–18]. Together with the already known two loop calculation of the jet function [25], this will allow us to reach a full $\mathcal{O}(\alpha_s^2)$ accuracy, at leading power, in describing the decay rates in general, and in extracting $|V_{ub}|$ in particular.

Beyond leading order in $\Lambda_{\rm QCD}/m_b$, one would expect the decay rate to factorize into sums of products of subleading hard functions and subleading jet functions convoluted with subleading shape functions. Of these power suppressed corrections, only the subleading shape functions were known. In this paper we have analyzed the subleading jet functions' contribution. These arise first at order $\mathcal{O}(\alpha_s)$ and appear in the partial rate convoluted with the leading order shape function.

First, we have argued that at order $\Lambda_{\rm QCD}/m_b$ only subleading jet and shape functions contribute. Subleading hard function can only appear when multiplied with subleading jet or shape functions. We have demonstrated this explicitly at one loop by showing that the $\mathcal{O}(\alpha_s)$ corrections which are $\Lambda_{\rm QCD}/m_b$ suppressed in the end point region, arise from two momentum regions: a hard-collinear region and a soft region. We have then shown that the soft region is accounted for in the parton model by the one loop matrix elements of the *known* "tree level" subleading shape functions. The hard-collinear region is accounted for in the parton model by the time ordered products of subleading SCET currents that apart from the heavy quark itself, do not include any soft fields or soft covariant derivatives.

In the main section of this paper, section 5, we have defined to all orders in $\alpha_s(\mu_i)$, for the case of a tree level hard function, the 8 subleading jet functions that can contribute to partial rates of inclusive B decays, and calculated their one loop expressions. After a short discussion of the renormalization of the subleading jet functions, we specialized to the phenomenologically interesting cases of $\bar{B} \to X_u\, l\, \bar{\nu}$ and the $Q_{7\gamma} - Q_{7\gamma}$ contribution to $\bar{B} \to X_s\, \gamma$ and presented explicit expressions for these decay modes. The main results of the paper are collected in section 5.3.

We can now summarize the factorization formula for inclusive B decays in the following way

$$d\Gamma \sim \overbrace{H \cdot J \otimes S + \frac{1}{m_b} \sum_i h \cdot J_0 \otimes s_i}^{\text{known}} + \overbrace{\frac{1}{m_b} \sum_k h \cdot j_k \otimes S}^{\text{new}} + \mathcal{O}\left(\frac{1}{m_b^2}\right),$$

where the label "new" refers to the new results of this paper. The label "known" refers to terms in the factorization formula for which we have *explicit* expressions for all of their perturbative components. Thus H is the leading order hard function, J is the leading order jet function, both known at $\mathcal{O}(\alpha_s^2)$, J_0 is the $\mathcal{O}(\alpha_s^0)$ part of the leading order jet function,

$h = 1 + \mathcal{O}(\alpha_s)$, and j_k are given in section 5.3. The rest of the $1/m_b$ suppressed terms for which we do not have such explicit expressions can be found in [26] and [28].

While the subleading jet functions' contribution is both α_s and $1/m_b$ suppressed in the end point region, their contribution becomes more important as one moves out of the this region, since the $1/m_b$ suppression is reduced as one is integrating over larger and larger portions of phase space. The *one-loop* subleading shape functions contribution, although formally α_s/m_b suppressed in the end point region, is expected to become even more power suppressed outside of the end point region. Furthermore, the kinematical area outside of the end point region is becoming more important with the constant improvement of experiments and the relaxation of kinematical cuts. Together with the recent calculation of the leading order hard function for semileptonic decays at $\mathcal{O}(\alpha_s^2)$, this work takes another step towards a more precise description of inclusive B decays in the end point region.

Although we have not presented any kind of numerical analysis, the implementation of the subleading jet function analysis with the framework of "BLNP" [29] is relatively easy. One needs to replace equation (2.13) by (5.46) and (2.14) by equations (5.48) and (5.49). At the same time one needs to modify the treatment of the subleading shape function in [29]. Only the combined numerical analysis would be meaningful. Such a study is left for future work.

Another issue that deserves further consideration is the renormalization of the subleading jet functions. In particular, it would be desirable to have a complete basis of subleading jet functions. As in the case of the subleading shape functions [49], it seems that further study is needed if we wish to understand the renormalization and mixing of these non local operators.

Finally, subleading jet functions arise also outside of flavor physics, for example, in the $x \to 1$ region of deep inelastic scattering. More specifically, the subleading jet function j_{11}^S appear in the factorization formula for the longitudinal structure function [33–36].[7] A more detailed study of the subleading jet functions' contributions to the $x \to 1$ region of deep inelastic scattering is left for future work.

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[7]Note though that the subleading jet function in [33, 34] and [35] differ from our definition and from each other.

A Consequences of PT symmetry

In this section we will prove our claim that PT symmetry and translation invariance allows us to relate the TOP of two different hard-collinear operators. Define

$$\begin{aligned} T_{ab} &= \int d^4x\, e^{-ipx} \langle\Omega|T\left\{O_a^\dagger(0), O_b(x)\right\}|\Omega\rangle = \\ &= \int d^4x\, e^{-ipx}\left[\theta(x^0)\langle\Omega|O_b(x)O_a^\dagger(0)|\Omega\rangle \pm \theta(-x^0)\langle\Omega|O_a^\dagger(0)O_b(x)|\Omega\rangle\right] \\ T_{ba} &= \int d^4x\, e^{-ipx} \langle\Omega|T\left\{O_b^\dagger(0), O_a(x)\right\}|\Omega\rangle = \\ &= \int d^4x\, e^{-ipx}\left[\theta(x^0)\langle\Omega|O_a(x)O_b^\dagger(0)|\Omega\rangle \pm \theta(-x^0)\langle\Omega|O_b^\dagger(0)O_a(x)|\Omega\rangle\right] \end{aligned} \tag{A.1}$$

We would like to relate T_{ab} and T_{ba}. Using translation invariance, and the PT invariance of the strong interactions and of the vacuum, we have

$$\begin{aligned} \langle\Omega|O_a(x)O_b^\dagger(0)|\Omega\rangle &= \langle\Omega|O_a(0)O_b^\dagger(-x)|\Omega\rangle = \left\langle\Omega\left|\left[\left(O_b^\dagger\right)^{PT}\right]^\dagger(x)\left[\left(O_a\right)^{PT}\right]^\dagger(0)\right|\Omega\right\rangle \\ \langle\Omega|O_b^\dagger(0)O_a(x)|\Omega\rangle &= \langle\Omega|O_b^\dagger(-x)O_a(0)|\Omega\rangle = \left\langle\Omega\left|\left[\left(O_a\right)^{PT}\right]^\dagger(0)\left[\left(O_b^\dagger\right)^{PT}\right]^\dagger(x)\right|\Omega\right\rangle. \end{aligned} \tag{A.2}$$

In order to relate T_{ab} and T_{ba} we need to relate

$$\begin{aligned} &\langle\Omega|\left[\left(O_b^\dagger\right)^{PT}\right]^\dagger(x)\left[\left(O_a\right)^{PT}\right]^\dagger(0)|\Omega\rangle \quad \text{to} \quad \langle\Omega|O_b(x)O_a^\dagger(0)|\Omega\rangle \\ &\langle\Omega|\left[\left(O_a\right)^{PT}\right]^\dagger(0)\left[\left(O_b^\dagger\right)^{PT}\right]^\dagger(x)|\Omega\rangle \quad \text{to} \quad \langle\Omega|O_a^\dagger(0)O_b(x)|\Omega\rangle. \end{aligned} \tag{A.3}$$

As our first example consider $\mathcal{J}_n$. In section 5 we defined,

$$\begin{aligned} \left(\frac{\not{n}}{2}\right)_{ab} \delta_{kl}\,\mathcal{J}_n(p^2) = \int d^4x\, e^{-ip\cdot x}\, &\Big[\theta(x^0)\langle\Omega|\left[\bar{\mathfrak{X}}\, n\cdot\mathcal{A}_{hc}\right]_{b\,l}(x)\mathfrak{X}_{a\,k}(0)|\Omega\rangle \\ &-\theta(-x^0)\langle\Omega|\mathfrak{X}_{a\,k}(0)\left[\bar{\mathfrak{X}}\, n\cdot\mathcal{A}_{hc}\right]_{b\,l}(x)|\Omega\rangle\Big], \end{aligned} \tag{A.4}$$

where k, l are color indices and a, b are spinor indices. Consider now the other TOP:

$$\begin{aligned} T_{\rm other} &= \int d^4x\, e^{-ip\cdot x}\langle\Omega|\,T\left\{\left[n\cdot\mathcal{A}_{hc}\,\mathfrak{X}\right]_{a\,l}(0),\, \bar{\mathfrak{X}}_{b\,k}(x)\right\}|\Omega\rangle = \\ =&\int d^4x\, e^{-ip\cdot x}\left[\theta(x^0)\langle\Omega|\bar{\mathfrak{X}}_{b\,k}(x)\left[n\cdot\mathcal{A}_{hc}\,\mathfrak{X}\right]_{a\,l}(0)|\Omega\rangle - \theta(-x^0)\langle\Omega|\left[n\cdot\mathcal{A}_{hc}\,\mathfrak{X}\right]_{a\,l}(0)\bar{\mathfrak{X}}_{b\,k}(x)|\Omega\rangle\right]. \end{aligned} \tag{A.5}$$

Using translation invariance and the PT symmetry we have, in the Weyl representation of Dirac γ matrices,

$$\begin{aligned} T_{\rm other} = \int d^4x\, e^{-ip\cdot x}\Big[&\theta(x^0)\langle\Omega|\left[\bar{\mathfrak{X}}\gamma^1\gamma^3\, n\cdot\mathcal{A}_{hc}\right]_{a\,l}(x)\left[\gamma^3\gamma^1\,\mathfrak{X}\right]_{b\,k}(0)|\Omega\rangle \\ &-\theta(-x^0)\langle\Omega|\left[\gamma^3\gamma^1\,\mathfrak{X}\right]_{b\,k}(0)\left[\bar{\mathfrak{X}}\gamma^1\gamma^3\, n\cdot\mathcal{A}_{hc}\right]_{a\,l}(x)|\Omega\rangle \Big]. \end{aligned} \tag{A.6}$$

Comparing (A.4) and (A.6) and using $\gamma^3 \gamma^1 \gamma^\mu \gamma^1 \gamma^3 = (\gamma^\mu)^T$ we find that

$$T_{\text{other}} = \left(\gamma^3 \gamma^1 \frac{\not{n}}{2} \gamma^1 \gamma^3\right)_{ba} \delta_{kl} \mathcal{J}_n(p^2) = \left(\frac{\not{n}}{2}\right)_{ab} \delta_{kl} \mathcal{J}_n(p^2), \tag{A.7}$$

which proves equation (5.4).

As a second example, consider $\mathcal{J}_{11}^S$ and $\mathcal{J}_{11}^A$. Here we use the same transformations to prove the decomposition the TOP into two jet functions. In section 5.1.3 we defined

$$\begin{aligned} T_{11} &= \int d^4x\, e^{-ip\cdot x} \langle\Omega| T\Big\{\Big[\mathcal{A}^\mu_{\perp hc}\, \mathfrak{X}\Big]_{a\,k}(0)\,, \Big[\bar{\mathfrak{X}}\, \mathcal{A}^\nu_{\perp\, hc}\Big]_{b\,l}(x)\Big\}|\Omega\rangle = \\ &\bar{n}\cdot p\, \frac{g_\perp^{\mu\nu}}{d-2} \delta_{kl}\, \mathcal{J}_{11}^S(p^2) \left(\frac{\not{n}}{2}\right)_{ab} + \bar{n}\cdot p\, \frac{i\epsilon_\perp^{\mu\nu}}{d-2} \delta_{kl}\, \mathcal{J}_{11}^A(p^2) \left(\frac{\not{n}}{2}\gamma_5\right)_{ab}. \end{aligned} \tag{A.8}$$

Using translation invariance and the PT symmetry we have

$$\begin{aligned} T_{11} &= \int d^4x\, e^{-ip\cdot x} \langle\Omega| T\Big\{\Big[\mathcal{A}^\nu_{\perp\, hc}\gamma^3\, \gamma^1\, \mathfrak{X}\Big]_{b\,l}(0), \Big[\bar{\mathfrak{X}}\, \gamma^1\, \gamma^3\, \mathcal{A}^\mu_{\perp hc}\Big]_{a\,k}(0)\Big\}|\Omega\rangle \\ &= \bar{n}\cdot p\, \frac{g_\perp^{\mu\nu}}{d-2} \delta_{kl}\, \mathcal{J}_{11}^S(p^2) \left(\gamma^3 \gamma^1 \frac{\not{n}}{2} \gamma^1 \gamma^3\right)_{ba} + \bar{n}\cdot p\, \frac{i\epsilon_\perp^{\nu\mu}}{d-2} \delta_{kl}\, \mathcal{J}_{11}^A(p^2) \left(\gamma^3 \gamma^1 \frac{\not{n}}{2} \gamma_5\, \gamma^1 \gamma^3\right)_{ba} \\ &= \bar{n}\cdot p\, \frac{g_\perp^{\mu\nu}}{d-2} \delta_{kl}\, \mathcal{J}_{11}^S(p^2) \left(\frac{\not{n}}{2}\right)_{ab} + \bar{n}\cdot p\, \frac{i\epsilon_\perp^{\mu\nu}}{d-2} \delta_{kl}\, \mathcal{J}_{11}^A(p^2) \left(\frac{\not{n}}{2}\gamma_5\right)_{ab}, \end{aligned} \tag{A.9}$$

which proves that $g_\perp^{\mu\nu}$ is accompanied by $(\not{n}/2)$, while $i\epsilon_\perp^{\mu\nu}$ is accompanied by $(\not{n}\gamma_5/2)$. In a similar manner we can use PT symmetry to analyze the rest of the TOPs.

B $T\{J^1, J^1\}$ subleading jet function(s)

In section 5.1.3 we argued that the TOP of the first order SCET current with itself, gives rise to only *one* subleading jet function. The reason is that the inverse derivative is acting on *all* the hard-collinear fields. On the other hand, we were forced to defined two different jet function j_n and $j_{n'}$ since the inverse derivative was only acting on the hard-collinear gluon and not on the hard-collinear quark. In this appendix we give a rigorous proof of these facts.

We consider the TOP of two currents: $J_1(x)$ and $\frac{1}{i\bar{n}\cdot\partial} J_2(x)$. Define

$$\begin{aligned} T_{12} &= i\int d^4x\, e^{-ip\cdot x} \Big\langle \Omega \Big| T\Big\{J_1^\dagger(0),\, J_2(x)\Big\}\Big|\Omega\Big\rangle \\ T_{21} &= i\int d^4x\, e^{-ip\cdot x} \Big\langle \Omega \Big| T\Big\{J_2^\dagger(0),\, J_1(x)\Big\}\Big|\Omega\Big\rangle \\ T_{12'} &= i\int d^4x\, e^{-ip\cdot x} \Big\langle \Omega \Big| T\Big\{J_1^\dagger(0),\, \frac{1}{i\bar{n}\cdot\partial}\, J_2(x)\Big\}\Big|\Omega\Big\rangle \\ T_{2'1} &= i\int d^4x\, e^{-ip\cdot x} \Big\langle \Omega \Big| T\Big\{J_2^\dagger(0)\, \frac{1}{-i\bar{n}\cdot\overleftarrow{\partial}},\, J_1(x)\Big\}\Big|\Omega\Big\rangle. \end{aligned} \tag{B.1}$$

We would like to prove that

$$\frac{1}{\pi}\,\text{Im}\Big(T_{12'} + T_{2'1}\Big) = -\frac{1}{\bar{n}\cdot p}\frac{1}{\pi}\,\text{Im}\Big(T_{12} + T_{12}\Big). \tag{B.2}$$

Consider $T_{12'}$ first. Inserting a complete set of states $|r\rangle$ we can write it as

$$T_{12'} = i \int d^4x\, e^{-ip\cdot x} \left[\theta(x^0) \sum_r \langle\Omega| \frac{1}{i\bar{n}\cdot\partial} J_2(x)|r\rangle\langle r|J_1^\dagger(0)|\Omega\rangle \right.$$
$$\left. \pm\theta(-x^0) \sum_r \langle\Omega|J_1^\dagger(0)|r\rangle\langle r| \frac{1}{i\bar{n}\cdot\partial} J_2(x)|\Omega\rangle \right]. \tag{B.3}$$

Using translation invariance we find

$$T_{12'} = i \int d^4x\, e^{-ip\cdot x} \left[\theta(x^0) \sum_r \frac{1}{\bar{n}\cdot r} \langle\Omega|J_2(0)|r\rangle\langle r|J_1^\dagger(0)|\Omega\rangle e^{-irx} \right.$$
$$\left. \pm\theta(-x^0) \sum_r \frac{1}{(-\bar{n}\cdot r)} \langle\Omega|J_1^\dagger(0)|r\rangle\langle r|J_2(0)|\Omega\rangle e^{irx} \right] \tag{B.4}$$

(we use r to denote both the state and its momentum). Using the identity

$$\theta(x^0) = \frac{i}{2\pi} \int d\omega\, \frac{e^{-i\omega x^0}}{\omega + i\epsilon},$$

and integrating over x and ω, we obtain

$$T_{12'} = i^2 \sum_r \langle\Omega|J_2(0)|r\rangle\langle r|J_1^\dagger(0)|\Omega\rangle \frac{1}{\bar{n}\cdot r} \frac{(2\pi)^3\delta^3(\vec{p}+\vec{r})}{-p_0 - r_0 + i\epsilon}$$
$$\pm i^2 \sum_r \langle\Omega|J_1^\dagger(0)|r\rangle\langle r|J_2(0)|\Omega\rangle \frac{1}{(-\bar{n}\cdot r)} \frac{(2\pi)^3\delta^3(\vec{p}-\vec{r})}{p_0 - r_0 + i\epsilon}. \tag{B.5}$$

Repeating the same procedure for $T_{2'1}$ we find

$$T_{2'1} = i^2 \sum_r \langle\Omega|J_1(0)|r\rangle\langle r|J_2^\dagger(0)|\Omega\rangle \frac{1}{\bar{n}\cdot r} \frac{(2\pi)^3\delta^3(\vec{p}+\vec{r})}{-p_0 - r_0 + i\epsilon}$$
$$\pm i^2 \sum_r \langle\Omega|J_1^\dagger(0)|r\rangle\langle r|J_2(0)|\Omega\rangle \frac{1}{(-\bar{n}\cdot r)} \frac{(2\pi)^3\delta^3(\vec{p}-\vec{r})}{p_0 - r_0 + i\epsilon}. \tag{B.6}$$

Since that by definition $\langle a|J_i|b\rangle^* = \langle b|J_i^\dagger|a\rangle$, we find that

$$T_{12'} + T_{2'1} = i^2 \sum_r 2\,\mathrm{Re}\left[\langle\Omega|J_2(0)|r\rangle\langle r|J_1^\dagger(0)|\Omega\rangle \frac{1}{\bar{n}\cdot r}\right] \frac{(2\pi)^3\delta^3(\vec{p}+\vec{r})}{-p_0 - r_0 + i\epsilon}$$
$$\pm i^2 \sum_r 2\,\mathrm{Re}\left[\langle\Omega|J_1^\dagger(0)|r\rangle\langle r|J_2(0)|\Omega\rangle \frac{1}{(-\bar{n}\cdot r)}\right] \frac{(2\pi)^3\delta^3(\vec{p}-\vec{r})}{p_0 - r_0 + i\epsilon}. \tag{B.7}$$

Using the identity $\mathrm{Im}\,[1/(u+i\epsilon)] = -\pi\,\delta(u)$, we have

$$\frac{1}{\pi}\,\mathrm{Im}\Big(T_{12'} + T_{2'1}\Big) = \frac{1}{(-\bar{n}\cdot p)} \sum_r 2\,\mathrm{Re}\left[\langle\Omega|J_2(0)|r\rangle\langle r|J_1^\dagger(0)|\Omega\rangle\right] (2\pi)^3\delta^4(p+r)$$
$$\pm\frac{1}{(-\bar{n}\cdot p)} \sum_r 2\,\mathrm{Re}\left[\langle\Omega|J_1^\dagger(0)|r\rangle\langle r|J_2(0)|\Omega\rangle\right] (2\pi)^3\delta^4(p-r)$$
$$= -\frac{1}{\bar{n}\cdot p}\frac{1}{\pi}\,\mathrm{Im}\Big(T_{12} + T_{12}\Big). \tag{B.8}$$

We should note that throughout the above derivation we have assumed that the $\bar{n}$ component of the total momentum of the state is not zero and therefore its inverse is defined. This assumption follows from the fact that by definition the hard-collinear states have a "large" $\bar{n}$ component of momentum. In the same way we can prove that

$$\frac{1}{\pi}\,\mathrm{Im}\left(i\int d^4x\,e^{-ip\cdot x}\langle\Omega|\,T\left\{J_2^\dagger(0)\,\frac{1}{-i\bar{n}\cdot\overleftarrow{\partial}},\,\frac{1}{i\bar{n}\cdot\partial}\,J_2(x)\right\}|\Omega\rangle\right)=$$
$$=\frac{1}{(\bar{n}\cdot p)^2}\frac{1}{\pi}\,\mathrm{Im}\left(i\int d^4x\,e^{-ip\cdot x}\langle\Omega|\,T\left\{J_2^\dagger(0),\,J_2(x)\right\}|\Omega\rangle\right). \qquad (B.9)$$

Using these identities we obtain the results of section 5.1.3.

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